Principal Curvature
Math 473
Introduction to Differential Geometry
Lecture 27

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Definition (2):
The **principal curvatures** of the surface X at a point p, denoted by κ_1 and κ_2, are the global maximum and the global minimum of the sectional curvature at the point p.

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Note
We have shown that principal curvatures correspond to eigenvalues and that principal directions $x \cdot X_u + y \cdot X_v$ correspond to eigenvectors (x, y) of the matrix $I^{-1} \cdot II$.

Recall:

The sectional curvature in a direction is the normal curvature of the curves in this direction. The normal curvature is given by $\kappa_n = \frac{1}{|\gamma'|} \cdot (T' \bullet N)$, hence $\kappa_n > 0$ if the curve is bending in the direction of N and $\kappa_n < 0$ if the curve is bending away from N.

Let κ_1 and κ_2 be the principal curvatures.

- If $\kappa_1, \kappa_2 > 0$, then the sectional curvature κ is positive in all directions. All sections of X at p are bending in the direction of N. In a neighbourhood of the point p the surface X is on the same side of its tangent plane as N.

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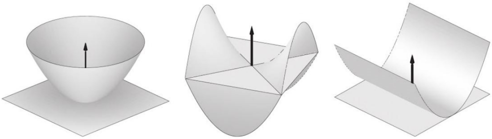
- If $\kappa_1, \kappa_2 < 0$, then the sectional curvature κ is negative in all directions. All sections of X at p are bending away from N. In a neighbourhood of the point p the surface X is on the other side of its tangent plane than N.
- If $\kappa_1 < 0, \kappa_2 > 0$ or $\kappa_1 > 0, \kappa_2 < 0$, then the sectional curvature is positive in some directions, negative in some directions and equal to zero in some directions. In a neighbourhood of the point p the points of the surface X are on both sides of its tangent plane and the surface looks like the saddle surface.

Definition (3): A point on a surface

- is **elliptic** if the principal curvatures are non-zero and of the same sign.
- is **hyperbolic** if the principal curvatures are non-zero and of different signs.
- is **parabolic** if at least one of the principal curvatures is equal to zero.



(a) Elliptic point (b) Hyperbolic point (c) Parabolic point

Examples (3):
All points on a sphere are elliptic. All points on the cylinder are parabolic. The origin on the saddle surface is hyperbolic.

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Remark:
In general, a surface can be subdivided into regions consisting of elliptic and hyperbolic points respectively, the boundaries between the regions are curves that consist of parabolic points.

Example (4):
Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $X(u, v) = (u, v, 1)$. Compute the first fundamental form of the surface X. Compute the second fundamental form of the surface X. Compute the principal curvatures of this surface. Is there umbilic point.

Example (5):
Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be given by $X(u,v) = (\cos u, \sin u, v)$. Compute the first fundamental form of the surface X. Compute the second fundamental form of the surface X. Compute the principal curvatures of this surface. Is there umbilic point.

Exercise (1):
Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be given by
$X(u, v) = (\sin u \sin v, \cos u \sin v, \cos v)$. Compute the first fundamental form of the surface X. Compute the second fundamental form of the surface X. Compute the principal curvatures of this surface. Is there umbilic point.

Thanks for listening.