Gauss Curvature and Mean Curvature
Math 473
Introduction to Differential Geometry
Lecture 29

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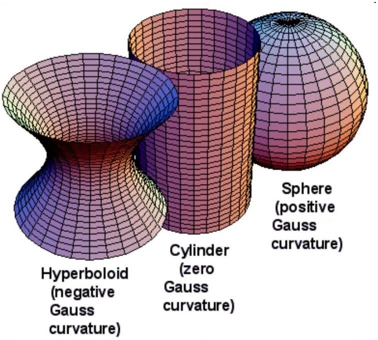
Definition (1):

Let κ_1 and κ_2 be the principal curvatures.
The **Gauss curvature** K of the surface X at a point p is the product of the principal curvatures

$$K = \kappa_1 \cdot \kappa_2.$$



Definition (2):
Let κ_1 and κ_2 be the principal curvatures.
The **mean curvature** H of the surface X at a point p is the average of the principal curvatures

$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$

Remark:
A point is

1. i) elliptic if and only if $K > 0$,
2. ii) hyperbolic if and only if $K < 0$,
3. iii) parabolic if and only if $K = 0$,

where K is the Gauss curvature of X at the point, i.e. the product of the principal curvatures.

Definition (2):

A surface patch is called **minimal** if it has mean curvature $H = 0$.

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Remark:
Minimal surfaces often have the property of having the smallest area for a surface with a given boundary.

Recall

Let

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be a (2×2)-matrix. Then, the trace of A is

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Proposition (1):

Let
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
be a (2×2)-matrix. Let λ_1 and λ_2 be the eigenvalues of the matrix A. Then
$$\lambda_1 + \lambda_2 = \text{trace}(A) = a + d, \quad \lambda_1 \cdot \lambda_2 = \det(A) = ad - bc.$$

Proposition (2):

Let $X : U \to \mathbb{R}^3$ be a regular surface patch and let p be a point on the surface X. Let

$$\mathrm{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{and} \quad \mathrm{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

be the matrices of the first and second fundamental form respectively at the point p.

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$$K = \det(\mathrm{I}^{-1} \cdot \mathrm{II}) = \frac{\det(\mathrm{II})}{\det(\mathrm{I})} = \frac{eg - f^2}{EG - F^2},$$

$$H = \frac{\operatorname{trace}(\mathrm{I}^{-1} \cdot \mathrm{II})}{2} = \frac{eG + gE - 2fF}{2(EG - F^2)}.$$

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Proof:

Example (1):
Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $X(u, v) = (u, v, uv)$. Compute the Gauss curvature and the mean curvature this surface at the point (0,0)? Is a typical point of the surface hyperbolic, parabolic or elliptic?

Example (2):
Show that the Gauss curvature and the mean curvature of the surface

$$X(u, v) = (u + v, u - v, uv)$$

at $u = v = 1$ are

$$K = -\frac{1}{16}, \quad H = \frac{1}{8\sqrt{2}}.$$

Is a typical point of the surface hyperbolic, parabolic or elliptic?

Thanks for listening.