Gauss-Weingarten Equations
Math 473
Introduction to Differential Geometry
Lecture 30

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Motivation:

Let $X : U \to \mathbb{R}^3$ be a regular surface patch and let $N : U \to \mathbb{R}^3$ be a unit normal on X. At any point of the surface we have the vectors X_u, X_v, N. The vectors X_u and X_v are linearly independent since the surface X is regular. The vector N is orthogonal to both X_u and X_v, hence is not a linear combination of X_u and X_v. Therefore the vectors X_u, X_v, N are linearly independent and form a basis.

The Gauss-Weingarten equations play the same role for the basis (X_u, X_v, N) as the Serret-Frenet equations play for the basis (T, N, B) and the Darboux equations play for the basis (T, U, N), namely the Gauss-Weingarten equations express the derivatives of the vectors X_u, X_v, N as linear combinations of the vectors X_u, X_v, N.

To be able to compute the coefficients in the Gauss-Weingarten equations, we need to compute the dot-products of the derivatives X_{uu}, X_{uv}, X_{vv}, N_u, N_v with the vectors X_u, X_v, N. To this end we consider the dot-products of the vectors X_u, X_v, N with each other

$$\begin{pmatrix} X_u \bullet X_u & X_u \bullet X_v & X_u \bullet N \\ X_v \bullet X_u & X_v \bullet X_v & X_v \bullet N \\ N \bullet X_u & N \bullet X_v & N \bullet N \end{pmatrix} = \begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and differentiate these equations.

Lemma (1):

Differentiating the equation $N \bullet N = 1$ with respect to u resp. v we obtain that

$$N_u \bullet N = N_v \bullet N = 0.$$

Thus the vectors N_u resp. N_v are orthogonal to the normal N (or equal to 0) and are therefore linear combinations of the vectors X_u and X_v.

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Lemma (2):

Differentiating the equations $X_u \bullet N = 0$ and $X_v \bullet N = 0$ we obtain that the coefficients of the second fundamental form satisfy the equations

$$\begin{aligned} e &= X_{uu} \bullet N = -X_u \bullet N_u, \\ f &= X_{uv} \bullet N = -X_u \bullet N_v = -X_v \bullet N_u, \\ g &= X_{vv} \bullet N = -X_v \bullet N_v. \end{aligned}$$

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Proof

Lemma (3):

Differentiating the equations

$$X_u \bullet X_u = E, \quad X_u \bullet X_v = X_v \bullet X_u = F, \quad X_v \bullet X_v = G$$

we obtain:

1. $X_{uu} \bullet X_u = \frac{1}{2} E_u,$
2. $X_{uu} \bullet X_v = F_u - \frac{1}{2} E_v,$
3. $X_{uv} \bullet X_u = \frac{1}{2} E_v,$
4. $X_{uv} \bullet X_v = \frac{1}{2} G_u,$
5. $X_{vv} \bullet X_u = F_v - \frac{1}{2} G_u,$
6. $X_{vv} \bullet X_v = \frac{1}{2} G_v.$

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4. $X_{uv} \bullet X_v = \frac{1}{2} G_u,$
5. $X_{vv} \bullet X_u = F_v - \frac{1}{2} G_u,$
6. $X_{vv} \bullet X_v = \frac{1}{2} G_v.$

Proof:

Theorem (1):

The Gauss-Weingarten equations express the partial derivatives X_{uu}, X_{uv}, X_{vv}, N_u, N_v in the basis X_u, X_v, N:

$$\begin{aligned} X_{uu} &= \Gamma^1_{11} \cdot X_u + \Gamma^2_{11} \cdot X_v + e \cdot N, \\ X_{uv} &= \Gamma^1_{12} \cdot X_u + \Gamma^2_{12} \cdot X_v + f \cdot N, \\ X_{vv} &= \Gamma^1_{22} \cdot X_u + \Gamma^2_{22} \cdot X_v + g \cdot N, \\ N_u &= \beta^1_1 \cdot X_u + \beta^2_1 \cdot X_v, \\ N_v &= \beta^1_2 \cdot X_u + \beta^2_2 \cdot X_v, \end{aligned}$$

with coefficients β^j_i and Γ^k_{ij}. The coefficients Γ^k_{ij} are called the *Christoffel symbols*.

The Christoffel symbols can be computed as follows

$$\begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{11} & \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} E_u/2 & E_v/2 & F_v - G_u/2 \\ F_u - E_v/2 & G_u/2 & G_v/2 \end{pmatrix}.$$

The coefficients β^j_i can be computed as follows

$$\begin{pmatrix} \beta^1_1 & \beta^1_2 \\ \beta^2_1 & \beta^2_2 \end{pmatrix} = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

Example (1):
Let $X : U \to \mathbb{R}^3$ be given by $X(u,v) = (u,v,1)$. Compute the Christoel symbols Γ_{ij}^k of X. Compute the coefficients β_i^j of X. Compute the Gauss-Weingarten equations of the surface X.

Example (2):
Let $X : U \to \mathbb{R}^3$ be given by $X(u, v) = (\cos v, \sin v, u)$. Compute the Christoel symbols Γ_{ij}^k of X. Compute the coefficients β_i^j of X. Compute the Gauss-Weingarten equations of the surface X.

Theorem (2): (Theorema Egregium)
The Gauss curvature is an intrinsic quantity, i.e. it can be expressed in terms of the coefficients E, F, G of the first fundamental form and their derivatives. The explicit formula for the Gauss curvature in terms of E, F, G is:

$$K = \frac{1}{(EG - F^2)^2} \cdot \left(\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix} \right).$$

Theorem (3): (Minding 1830)
The geodesic curvature of a curve on a surface is an intrinsic quantity. The explicit formula for the geodesic curvature of the curve $\gamma(t) = X(u(t), v(t))$ is

$$\kappa_g = \frac{\sqrt{EG - F^2}}{|\gamma'|^3} \cdot \begin{vmatrix} u' & u'' + \Gamma^1_{11} \cdot u'^2 + 2\Gamma^1_{12} \cdot u'v' + \Gamma^1_{22} \cdot v'^2 \\ v' & v'' + \Gamma^2_{11} \cdot u'^2 + 2\Gamma^2_{12} \cdot u'v' + \Gamma^2_{22} \cdot v'^2 \end{vmatrix},$$

where $|\gamma'| = \sqrt{u'^2 \cdot E + 2u'v' \cdot F + v'^2 \cdot G}$.

Exercise (1):
Let $X:\mathbb{R}^2\to\mathbb{R}^3$ be given by
$X(u,v)=(\sin u\sin v,\cos u\sin v,\cos u)$. Compute the Christoel symbols Γ_{ij}^k of X. Compute the coefficients β_i^j of X. Compute the Gauss-Weingarten equations of the surface X.

Thanks for listening.