Serret-Frenet Equations for Space Curve
Math 473
Introducation to Differential Geometry
Lecture 7

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Theorem (1):
For a regular parametrised space curve $\alpha : I \mapsto \mathbb{R}^3$ the following Serret-Frenet equations are satisfied (with $S = |\alpha'|$):

$$\begin{aligned} T' &= \kappa \cdot s \cdot N, \\ N' &= -\kappa \cdot s \cdot T + \tau \cdot s \cdot B, \\ B' &= -\tau \cdot s \cdot N. \end{aligned}$$

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For a unit speed curve, we have $s = |\alpha'| = 1$ and the Serret-Frenet equations have the following form:

$$\begin{aligned} T' &= \kappa \cdot N, \\ N' &= -\kappa \cdot T + \tau \cdot B, \\ B' &= -\tau \cdot N. \end{aligned}$$

Remark(1): The Serret-Frenet equations can be written in the following matrix form

$$\begin{pmatrix} T' & N' & B' \end{pmatrix} = |\alpha'| \cdot \begin{pmatrix} T & N & B \end{pmatrix} \cdot \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

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Remark(2): For a unit speed curve we have $s = |\alpha'| = 1$ and the Serret-Frenet equations can be written as

$$\begin{pmatrix} T' & N' & B' \end{pmatrix} = \begin{pmatrix} T & N & B \end{pmatrix} \cdot \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

Proof of Theorem (for α is a unit speed):

Definition(1): Plane Curve
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Definition(2):

Let $\alpha : I \mapsto \mathbb{R}^3$ be a regular curve. We say α is a plane curve if there is unit vector such that

$$\overrightarrow{(\alpha(t) - \alpha(0))} \bullet \vec{u} = 0$$

Theorem (2):
Let $\alpha : I \mapsto \mathbb{R}^3$ be a unit speed curve with $\kappa > 0$ and S is the arc-length. Then, α is a plane curve if and only if $\tau(t) = 0$ for all $t \in I$.

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Let $\alpha : I \mapsto \mathbb{R}^3$ be a unit speed curve with $\kappa > 0$ and S is the arc-length. Then, α is a plane curve if and only if $\tau(t) = 0$ for all $t \in I$.

Proof:

Thanks for listening.