Engineering Mechanics
AGE 2330

Lect 3: Moment

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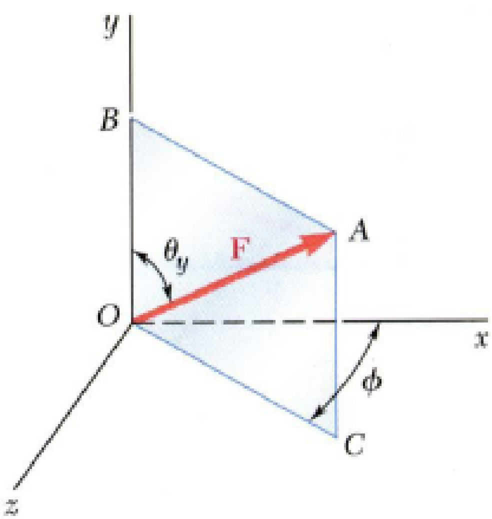
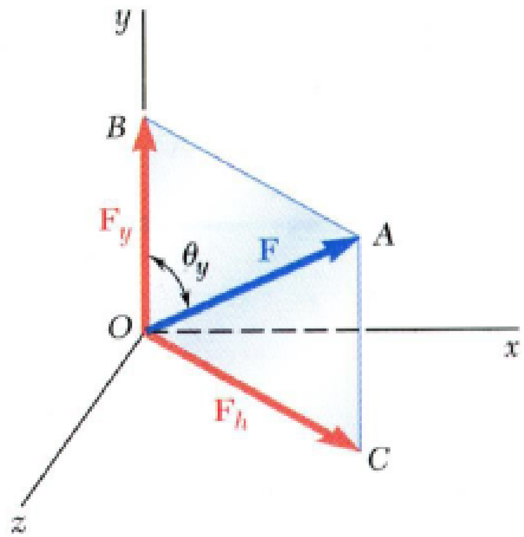
Rectangular Components in Space



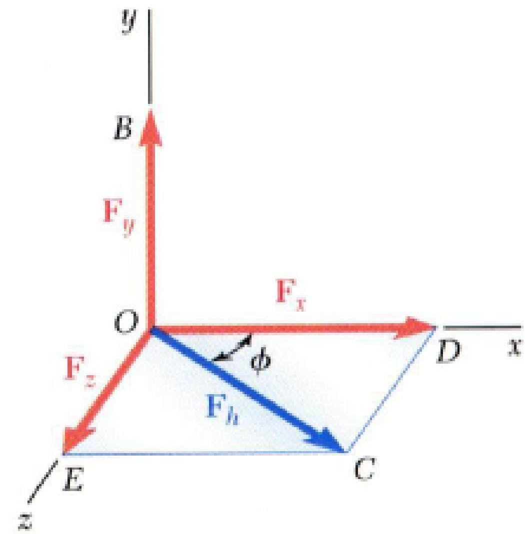
- Vector $\mathbf{F}$ is contained in the plane *OBAC*



- Resolve $\mathbf{F}$ into horizontal and vertical components.

$$F_y = F\cos\theta_y$$

$$F_h = F\sin\theta_y$$

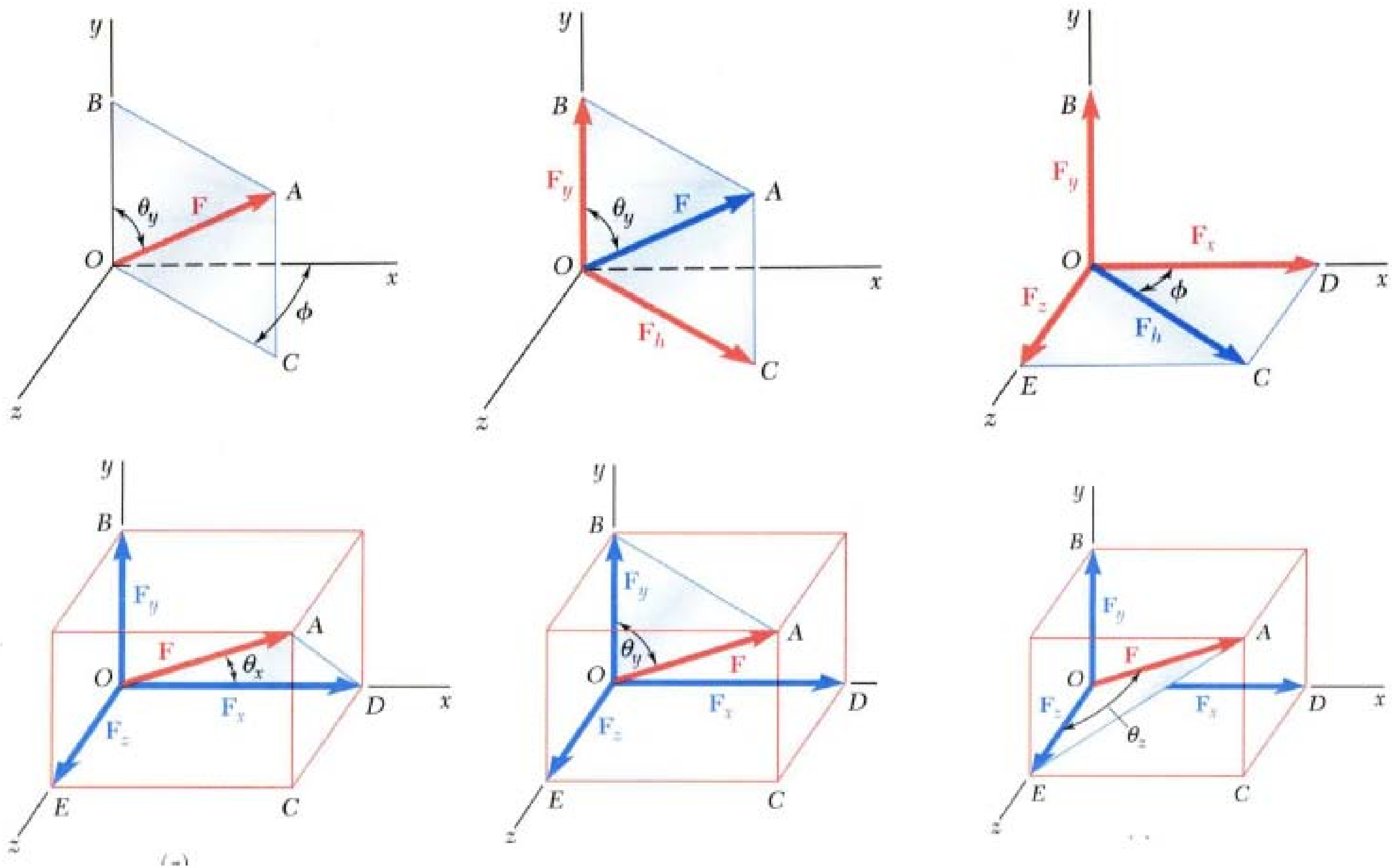


- Resolve F_h into rectangular components

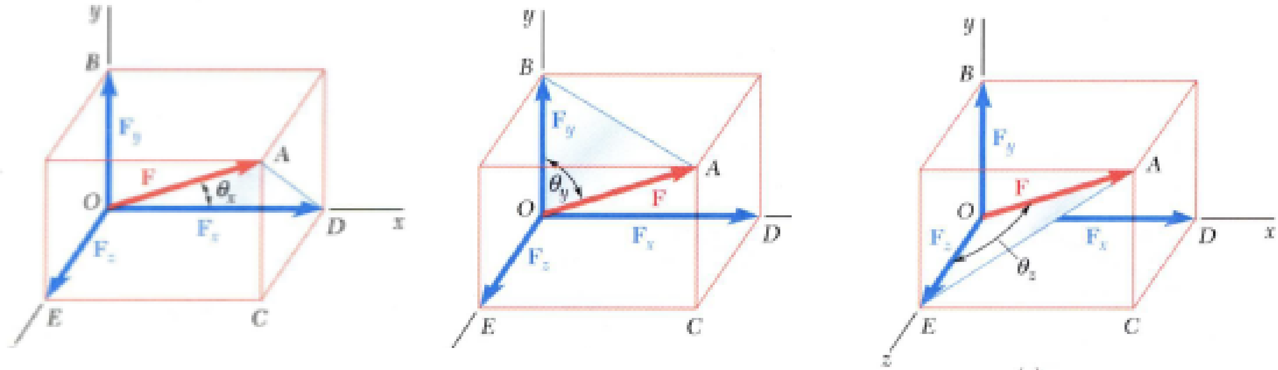
$$\begin{aligned} F_x &= F_h\cos\phi \\ &= F\sin\theta_y\cos\phi \end{aligned}$$

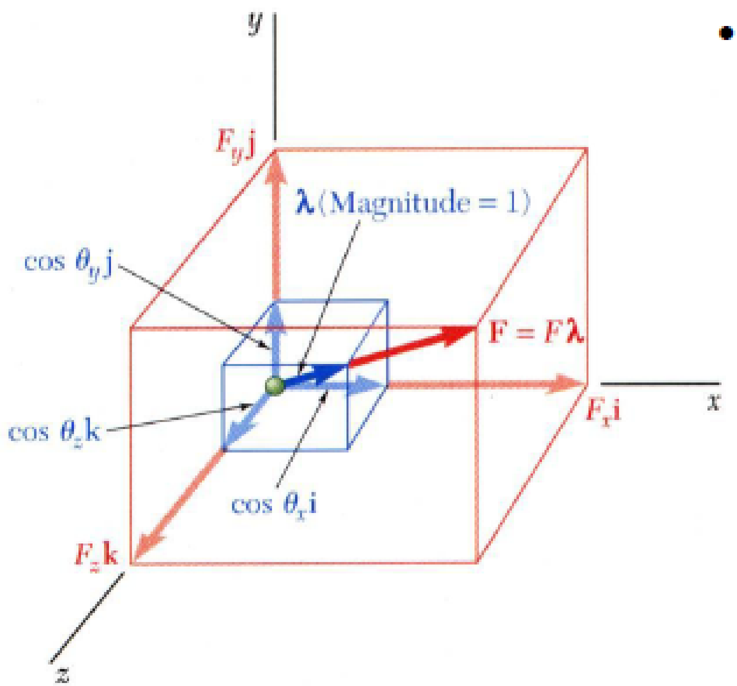
$$\begin{aligned} F_z &= F_h\sin\phi \\ &= F\sin\theta_y\sin\phi \end{aligned}$$

Spatial Components (*Direction Cosines*)



Spatial Components (*Direction Cosines*)





- With the angles between **F** and the axes,

$$F_x = F\cos\theta_x \quad F_y = F\cos\theta_y \quad F_z = F\cos\theta_z$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$= F\left(\cos\theta_x\vec{i} + \cos\theta_y\vec{j} + \cos\theta_z\vec{k}\right)$$

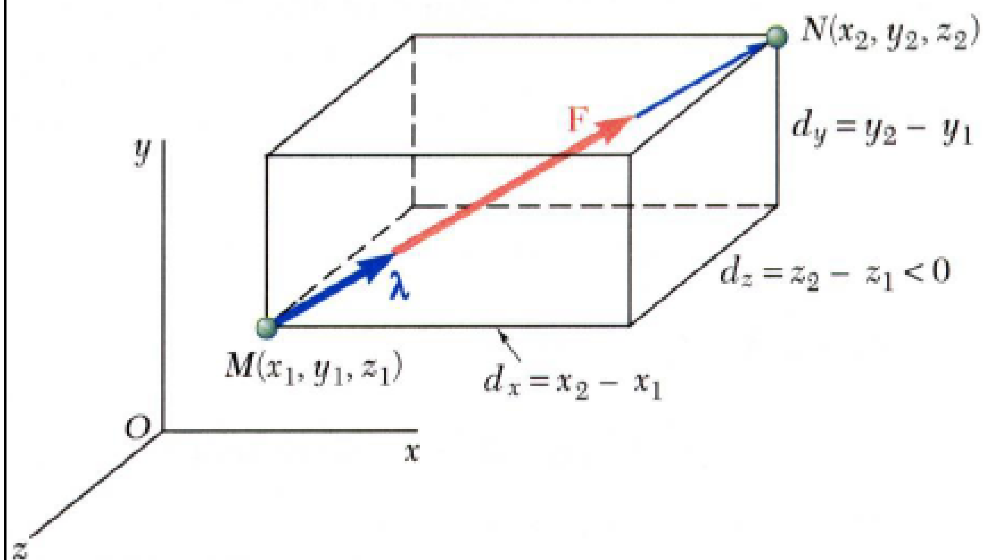
$$= F\vec{\lambda}$$

$$\vec{\lambda} = \cos\theta_x\vec{i} + \cos\theta_y\vec{j} + \cos\theta_z\vec{k}$$

- $\vec{\lambda}$ is a **unit vector** along the line of action of **F**; $\cos\theta_x$, $\cos\theta_y$, and $\cos\theta_z$ are the **direction cosines**

Rectangular Components in Space

- **Direction of force F**
  - Defined by **location** of **two points**
    - *M* (x_1, y_1 and z_1) and *N* (x_2, y_2 and z_2)



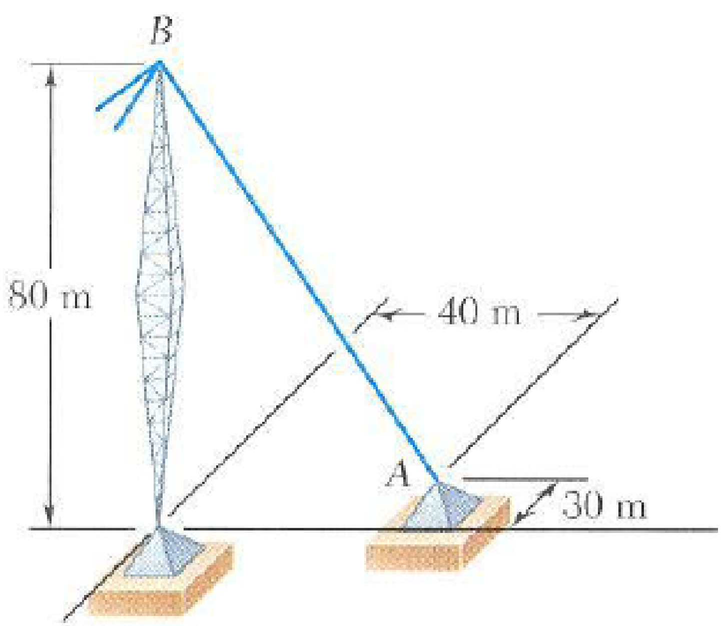
$$\vec{d} = \text{vector joining } M \text{ and } N$$
$$= d_x\vec{i} + d_y\vec{j} + d_z\vec{k}$$
$$d_x = x_2 - x_1 \quad d_y = y_2 - y_1 \quad d_z = z_2 - z_1$$
$$\vec{F} = F\vec{\lambda}$$
$$\vec{\lambda} = \frac{1}{d}\left(d_x\vec{i} + d_y\vec{j} + d_z\vec{k}\right)$$
$$F_x = \frac{Fd_x}{d} \quad F_y = \frac{Fd_y}{d} \quad F_z = \frac{Fd_z}{d}$$

Rectangular Components in Space

Example: The tension in the guy wire is 2500 N. Determine:

a) components F_x, F_y, F_z of the force acting on the bolt at *A*,

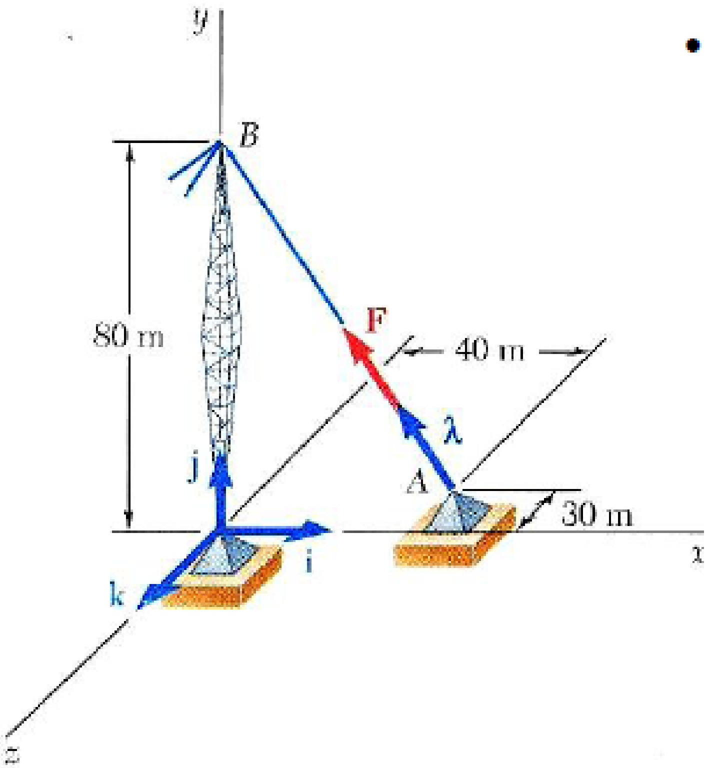
b) the angles q_x, q_y, q_z defining the direction of the force



SOLUTION:

- Based on the relative locations of the points *A* and *B*, determine the unit vector pointing from *A* towards *B*.
- Apply the unit vector to determine the components of the force acting on *A*.
- Noting that the components of the unit vector are the direction cosines for the vector, calculate the corresponding angles.

Rectangular Components in Space



Solution

- Determine the unit vector pointing from A towards B.

$$\overline{AB} = (-40\,\text{m})\vec{i} + (80\,\text{m})\vec{j} + (30\,\text{m})\vec{k}$$

$$AB = \sqrt{(-40\,\text{m})^2 + (80\,\text{m})^2 + (30\,\text{m})^2}$$

$$= 94.3\,\text{m}$$

$$\vec{\lambda} = \left(\frac{-40}{94.3}\right)\vec{i} + \left(\frac{80}{94.3}\right)\vec{j} + \left(\frac{30}{94.3}\right)\vec{k}$$

$$= -0.424\vec{i} + 0.848\vec{j} + 0.318\vec{k}$$

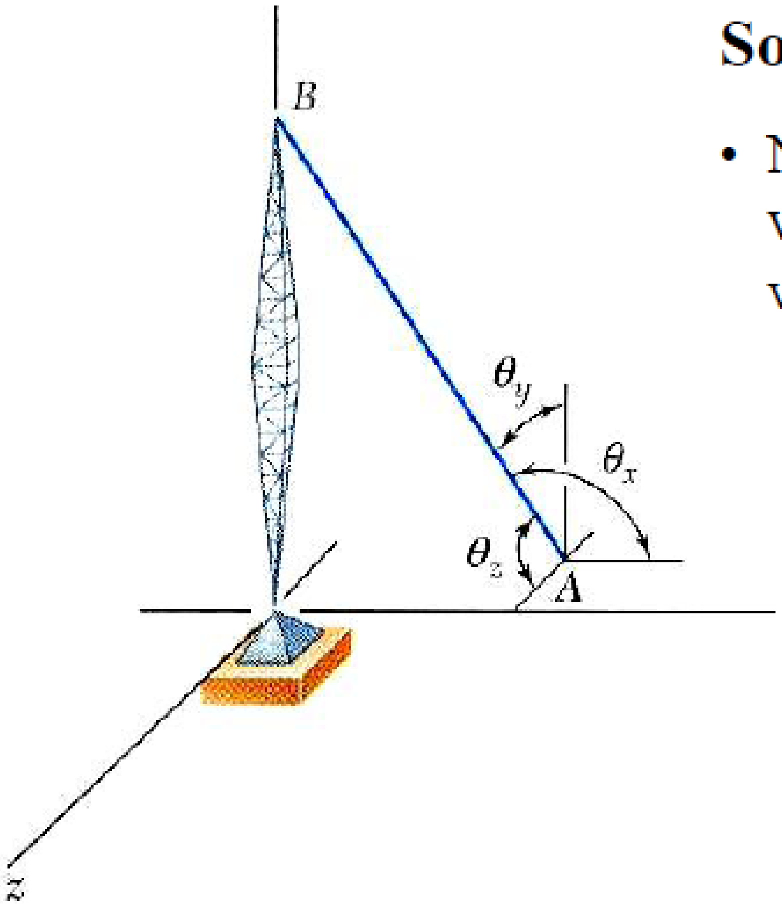
- Determine the components of the force.

$$\vec{F} = F\vec{\lambda}$$

$$= (2500\,\text{N})\left(-0.424\vec{i} + 0.848\vec{j} + 0.318\vec{k}\right)$$

$$= (-1060\,\text{N})\vec{i} + (2120\,\text{N})\vec{j} + (795\,\text{N})\vec{k}$$

Rectangular Components in Space



Solution

- Noting that the components of the unit vector are the direction cosines for the vector, calculate the corresponding angles.

$$\vec{\lambda} = \cos\theta_x \vec{i} + \cos\theta_y \vec{j} + \cos\theta_z \vec{k}$$

$$= -0.424\vec{i} + 0.848\vec{j} + 0.318\vec{k}$$

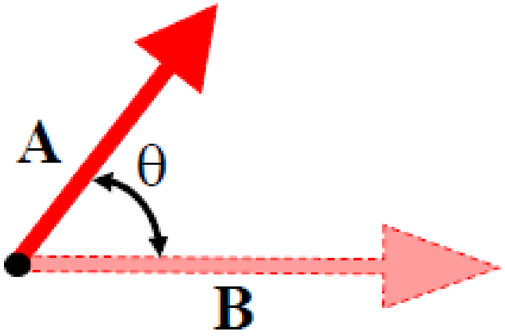
$$\theta_x = 115.1^\circ$$

$$\theta_y = 32.0^\circ$$

$$\theta_z = 71.5^\circ$$

Product of 2 Vectors: Dot Product

- **Dot Product** *(Scalar product)*
  - $\mathbf{A}.\mathbf{B} = AB \cos \theta$



  - **Applications**
    - Determination of the angle between two vectors

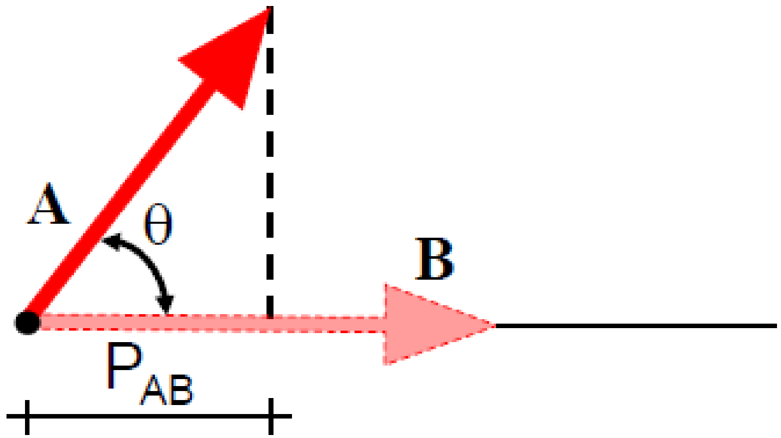
$$\mathbf{A}.\mathbf{B} = (A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}).(B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k}) = A_xB_x+A_yB_y+A_zB_z$$

$$\mathbf{A}.\mathbf{B} = AB\cos\theta$$

Obtain θ

Product of 2 Vectors: Dot Product

- **Applications**
  - Determination of the projection of a vector on a given axis

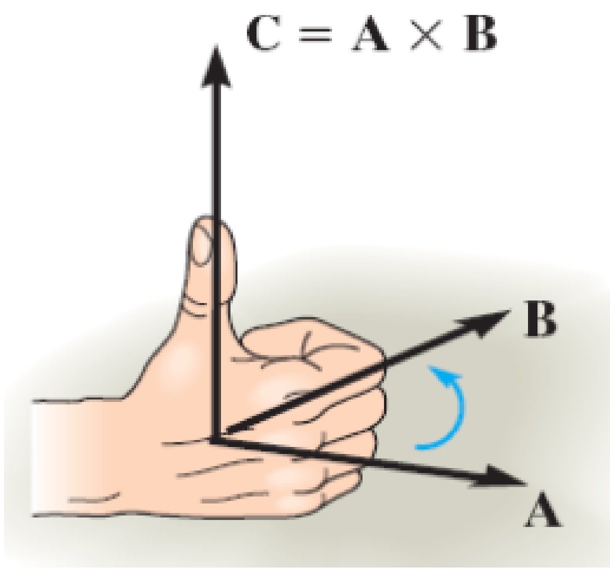


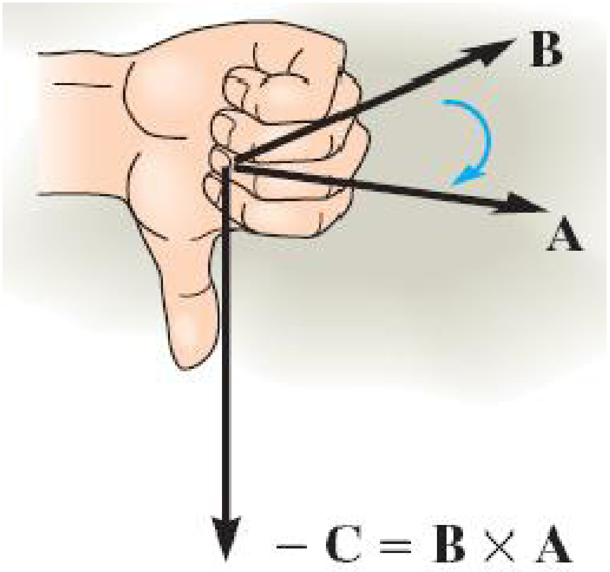
$$\mathbf{A}.\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

$$P_{AB} = A\cos\theta = (\mathbf{A}.\mathbf{B})/B$$

Product of 2 Vectors: Cross Product

- **Cross Product** *(Vector Product)*
  - $\mathbf{C} = \mathbf{A} \times \mathbf{B}$
    - $C = AB \sin \theta$

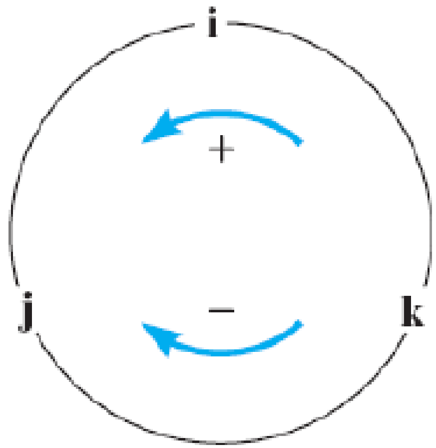


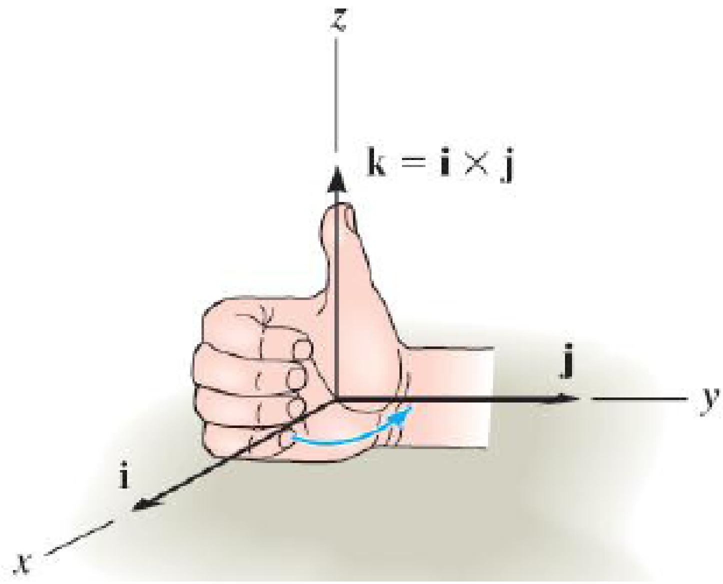


Product of 2 Vectors: Cross Product

- **Cross Product**

Cartesian Vector





$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \mathbf{i} \times \mathbf{i} = 0$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{j} \times \mathbf{j} = 0$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \mathbf{k} \times \mathbf{k} = 0$$

Product of 2 Vectors: Cross Product

- Cross Product
  - Distributive property
    - $\mathbf{C} \times (\mathbf{A}+\mathbf{B}) = \mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B}$

$$\mathbf{A}\times\mathbf{B} = (A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}) \times (B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k})$$

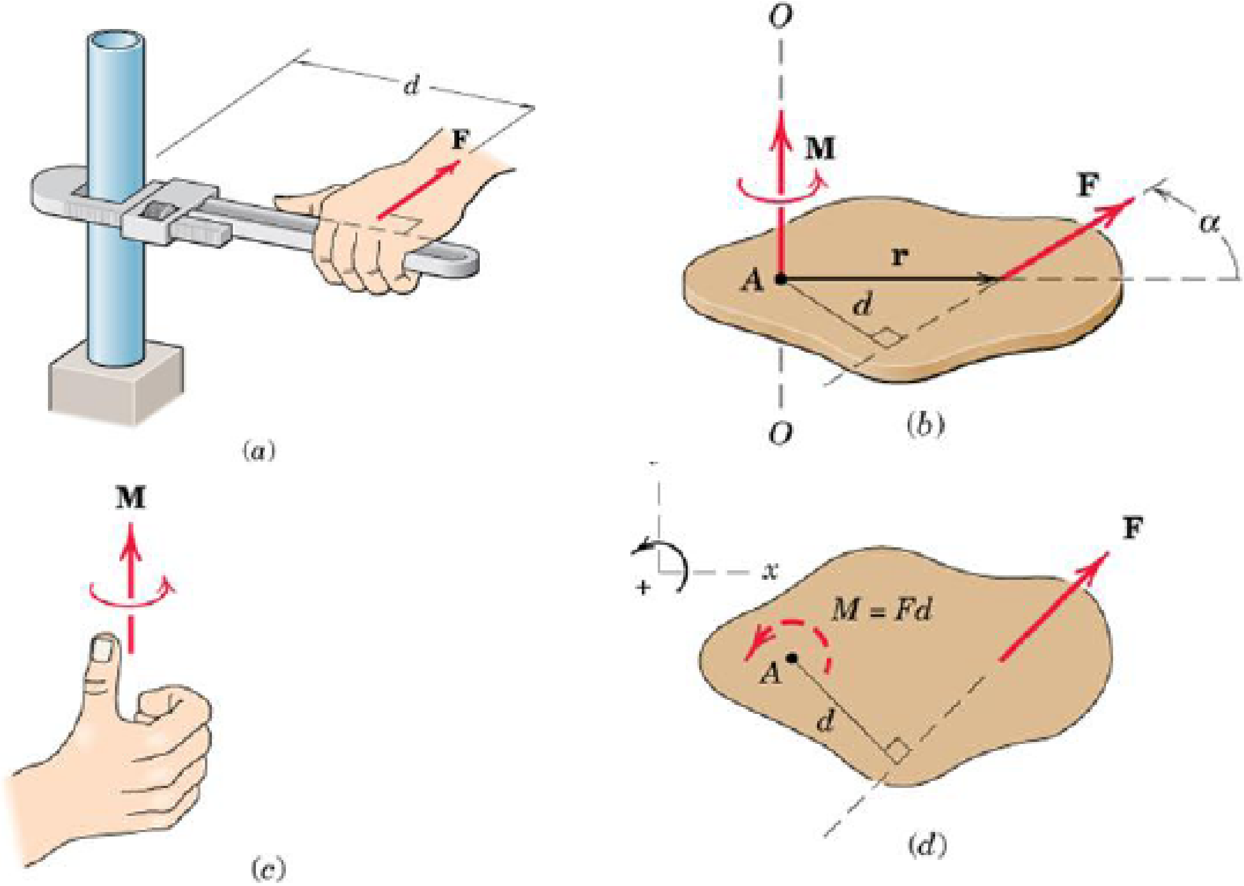
$$= (A_yB_z - A_zB_y)\mathbf{i} + (\ldots.)\mathbf{j} + (\ldots.)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_X & A_Y & A_Z \\ B_X & B_Y & B_Z \end{vmatrix}$$

$$\begin{array}{lll} \mathbf{i}\times\mathbf{j}=\mathbf{k} & \mathbf{i}\times\mathbf{k}=-\mathbf{j} & \mathbf{i}\times\mathbf{i}=\mathbf{0} \\ \mathbf{j}\times\mathbf{k}=\mathbf{i} & \mathbf{j}\times\mathbf{i}=-\mathbf{k} & \mathbf{j}\times\mathbf{j}=\mathbf{0} \\ \mathbf{k}\times\mathbf{i}=\mathbf{j} & \mathbf{k}\times\mathbf{j}=-\mathbf{i} & \mathbf{k}\times\mathbf{k}=\mathbf{0} \end{array}$$

Moment of a Force (Torque)

- **Moment** of a **Force (F) @ point A**
  - $\mathbf{M_o} = \mathbf{r} \times \mathbf{F}$

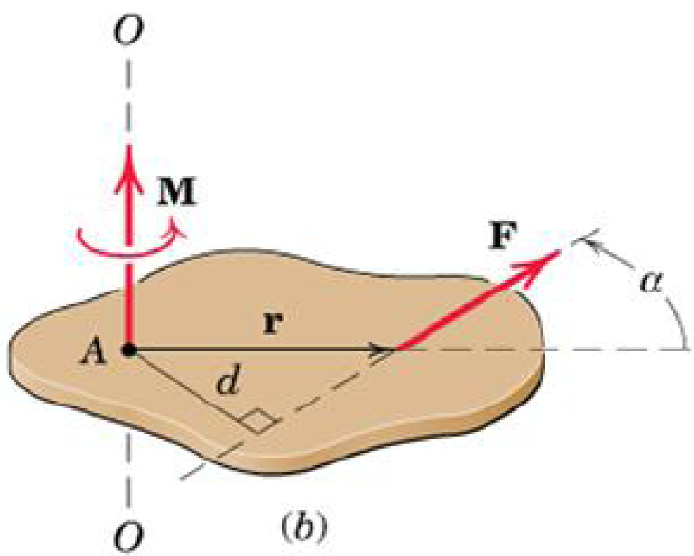


r = position vector directed from O to any point on the line of action of **F**

Moment of a Force

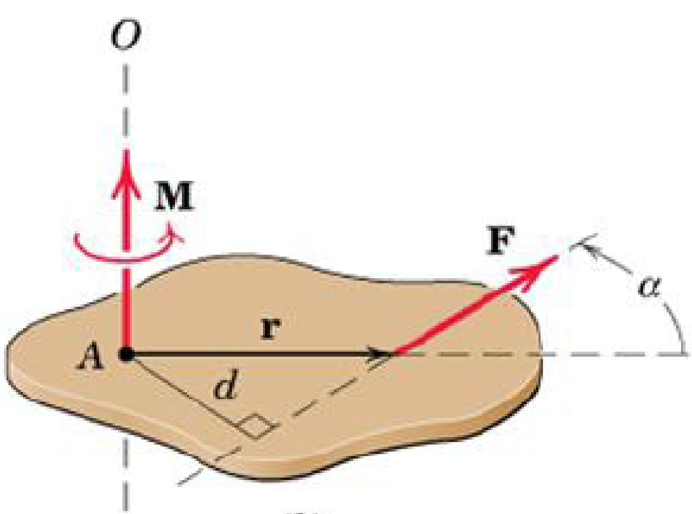
- **Magnitude** of **Moment**
  - tendency of $\mathbf{F}$ to cause rotation of the body about an axis along $\mathbf{M_O}$

$$M_O = rF\sin\alpha = F(r\sin\alpha) = Fd$$



Moment of a Force

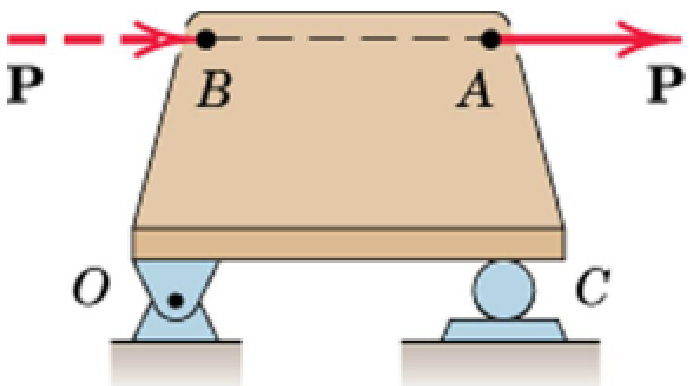
- Characteristic
  - Moment arm ($d = r\,sin\alpha$) does not depend on the particular point on the line of action of **F** to which the vector **r** is directed
    - **Sliding vector**
      - Line of action same as the moment axis

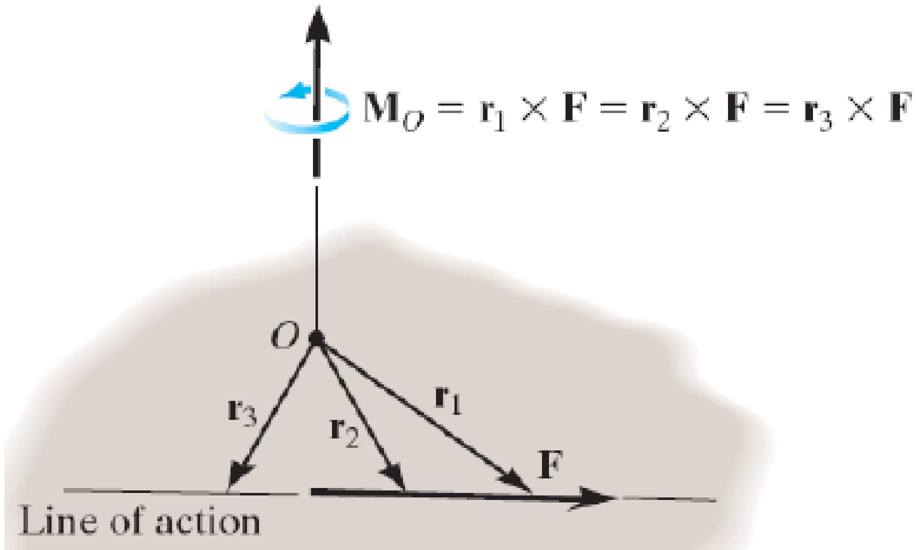


(b)

Moment of a Force

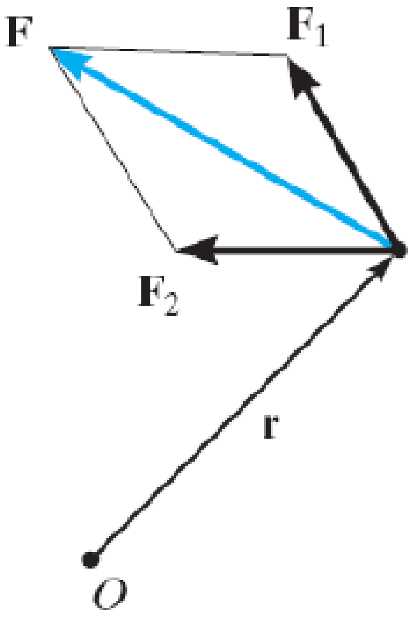
- **Principle of Transmissibility**
  - **Two forces** are **equivalent**
    - **Same magnitude, direction** and **lines of action**
    - **Same magnitude, direction** and **equal moments** about a **given point**





Moment of a Force

- **Varignon's Theorem**
  - **Moment** of a **force** about **a point** is equal to the **sum of the moments** of the **components of the force** about the **same point**

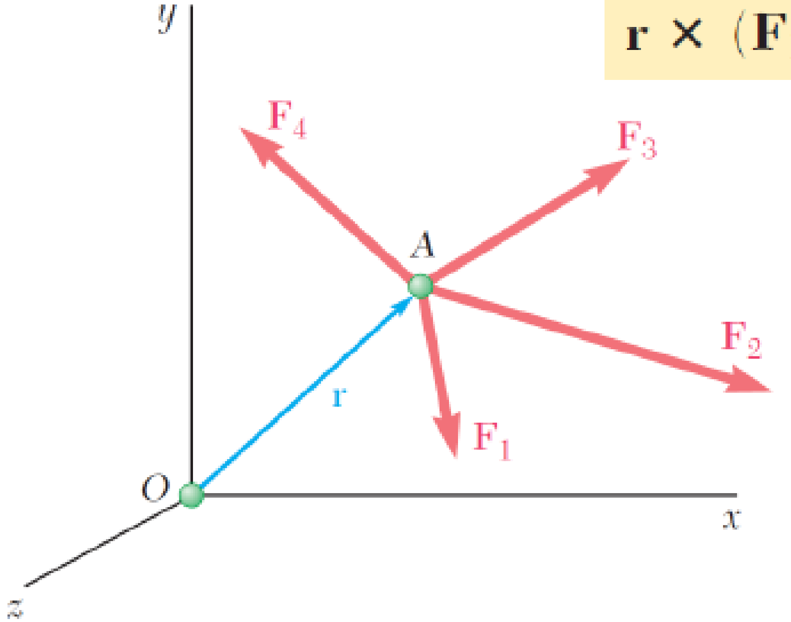


$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

Moment of a System of Concurrent Forces

- **Varignon's Theorem**
  - **Moment** of the **resultant** of a system of **concurrent forces** about **a point** is equal to the **sum of the moments** of the **of the individual forces** about the **same point**

$$\mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \cdots) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 + \cdots$$



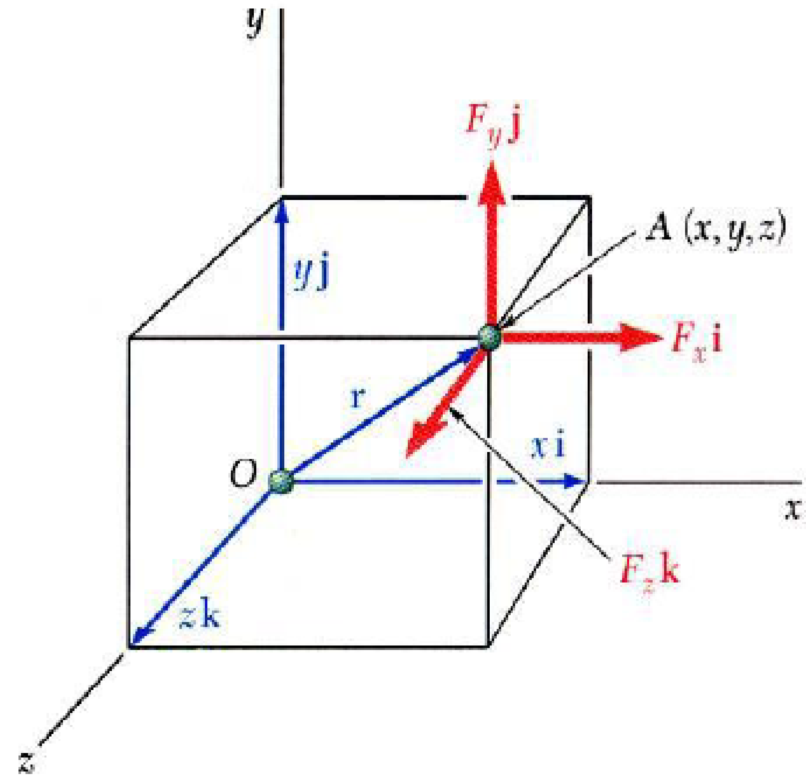
Rectangular Components of Moment

The moment of **F** about *O*,

$$\vec{M}_O = \vec{r} \times \vec{F}, \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$\vec{M}_O = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}$$



$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(yF_z - zF_y\right)\vec{i} + \left(zF_x - xF_z\right)\vec{j} + \left(xF_y - yF_x\right)\vec{k}$$

Rectangular Components of Moment

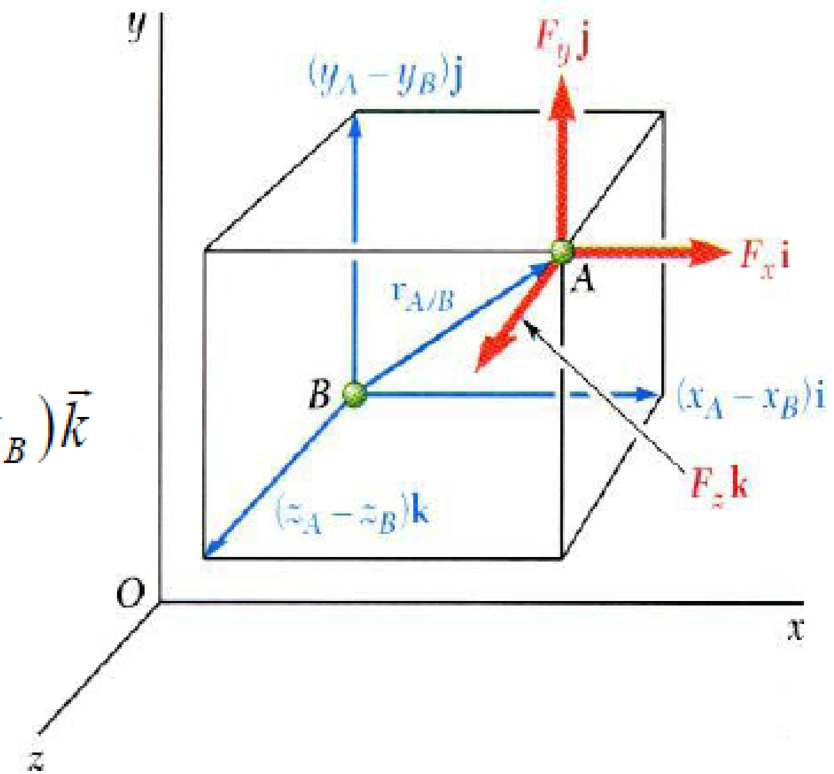
The moment of **F** about *B*,

$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F}$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

$$= (x_A - x_B)\vec{i} + (y_A - y_B)\vec{j} + (z_A - z_B)\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$



$$\vec{M}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{vmatrix}$$

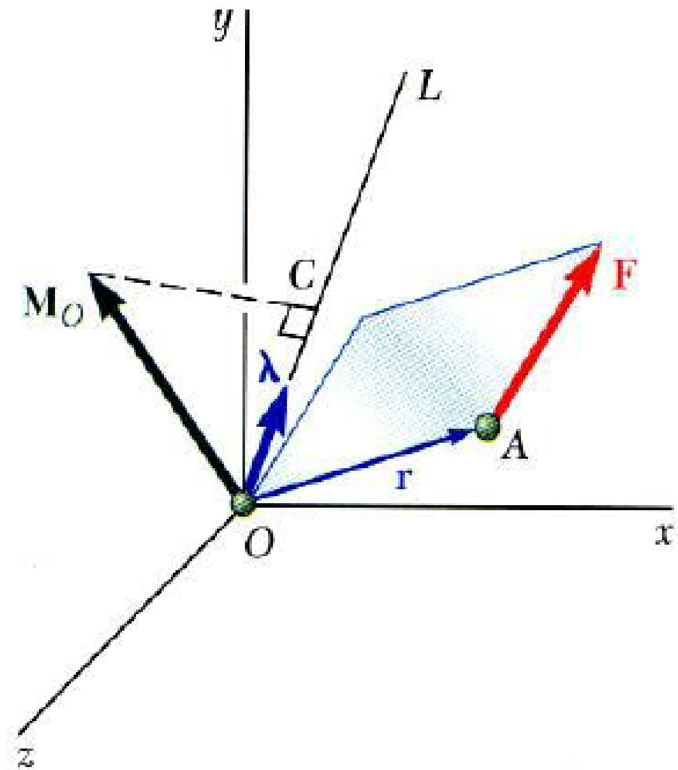
Moment of a Force About a Given Axis

- Moment $\mathbf{M_O}$ of a force $\mathbf{F}$ applied at the point $\mathbf{A}$ about a point $\mathbf{O}$,

$$\vec{M}_O = \vec{r} \times \vec{F}$$

- **Scalar moment** M_{OL} about an axis OL is the **projection** of the **moment vector** $\mathbf{M_O}$ onto the **axis**,

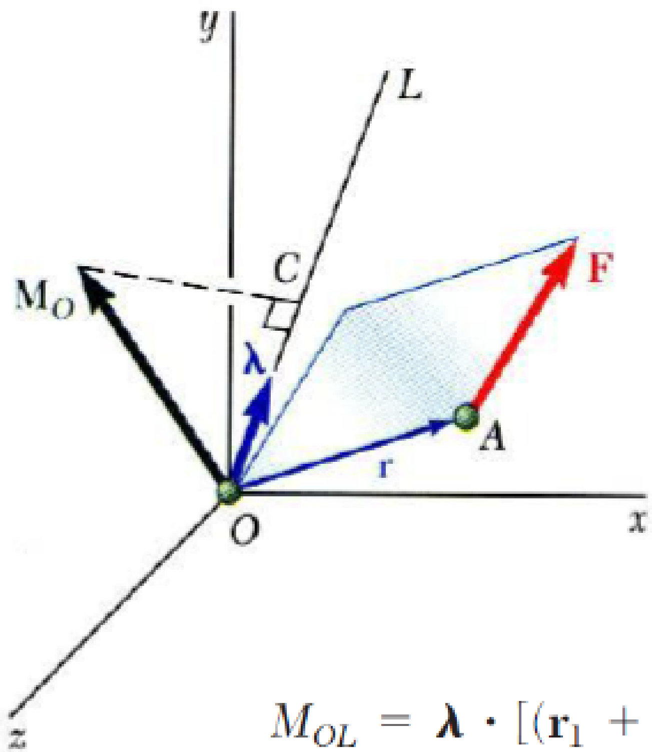
$$M_{OL} = \vec{\lambda} \bullet \vec{M}_O = \vec{\lambda} \bullet \left(\vec{r} \times \vec{F}\right)$$

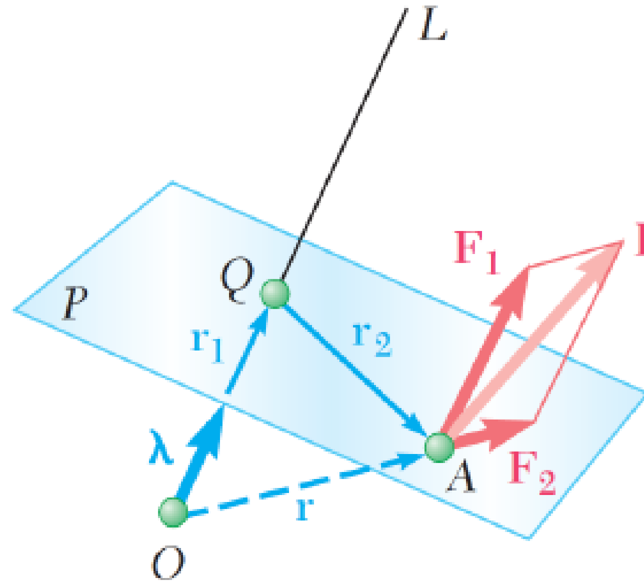


Application of Scalar Product

Moment of a Force About a Given Axis

- Significance of M_{OL}
  - Resolve into components





$$
\begin{aligned}
M_{OL} &= \boldsymbol{\lambda} \cdot [(\mathbf{r}_1 + \mathbf{r}_2) \times (\mathbf{F}_1 + \mathbf{F}_2)] \\
&= \cancel{\boldsymbol{\lambda} \cdot (\mathbf{r}_1 \times \mathbf{F}_1)} + \cancel{\boldsymbol{\lambda} \cdot (\mathbf{r}_1 \times \mathbf{F}_2)} + \cancel{\boldsymbol{\lambda} \cdot (\mathbf{r}_2 \times \mathbf{F}_1)} + \boldsymbol{\lambda} \cdot (\mathbf{r}_2 \times \mathbf{F}_2) \\
&= \boldsymbol{\lambda} \cdot (\mathbf{r}_2 \times \mathbf{F}_2)
\end{aligned}
$$

Moment of a Force About a Given Axis

- Significance of M_{OL}
  - M_{OL} is a measure of the **tendency** of the $\mathbf{F}$ to impart a **rigid body rotation** about the axis OL

$$M_{OL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

- Moments of $\mathbf{F}$ about the coordinate axes

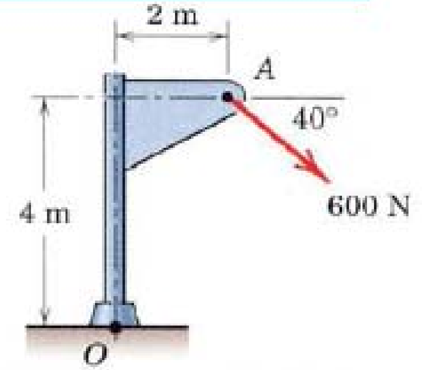
$$M_x = yF_z - zF_y$$

$$M_y = zF_x - xF_z$$

$$M_z = xF_y - yF_x$$

Moment: Example

Calculate the magnitude of the moment about the base point O of the 600-N force in five different ways.

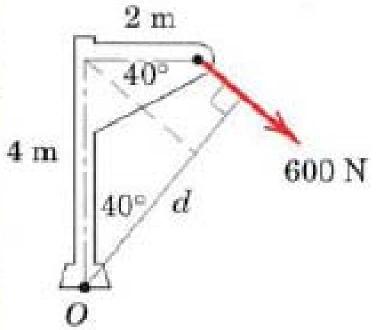


Solution. (*I*) The moment arm to the 600-N force is

$$d = 4 \cos 40^\circ + 2 \sin 40^\circ = 4.35 \text{ m}$$

By $M = Fd$ the moment is clockwise and has the magnitude

$$M_O = 600(4.35) = 2610 \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

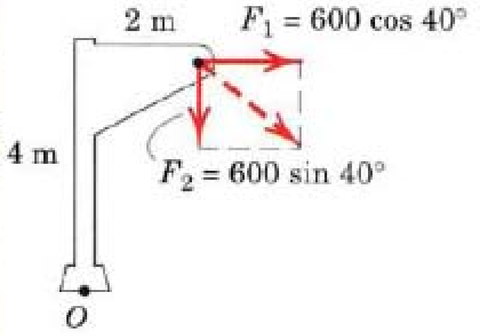


(*II*) Replace the force by its rectangular components at A

$$F_1 = 600 \cos 40^\circ = 460 \text{ N}, \qquad F_2 = 600 \sin 40^\circ = 386 \text{ N}$$

By Varignon's theorem, the moment becomes

$$M_O = 460(4) + 386(2) = 2610 \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$



(*III*) By the principle of transmissibility, move the 600-N force along its line of action to point B, which eliminates the moment of the component F_2. The moment arm of F_1 becomes

$$d_1 = 4 + 2 \tan 40^\circ = 5.68 \text{ m}$$

Moment: Example

and the moment is

$$M_O = 460(5.68) = 2610 \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

(IV) Moving the force to point C eliminates the moment of the component F_1. The moment arm of F_2 becomes

$$d_2 = 2 + 4 \cot 40^\circ = 6.77 \text{ m}$$

and the moment is

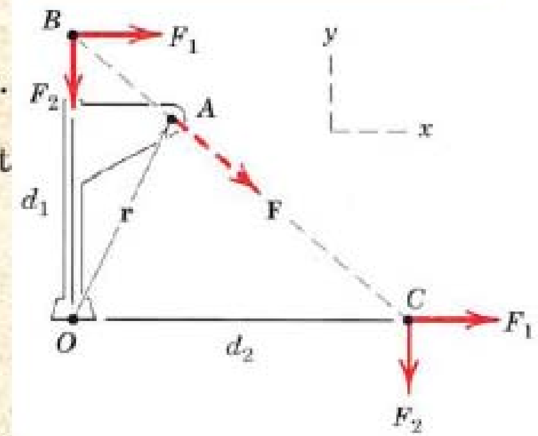
$$M_O = 386(6.77) = 2610 \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

(V) By the vector expression for a moment, and by using the coordinate system indicated on the figure together with the procedures for evaluating cross products, we have

$$\begin{aligned} \mathbf{M}_O &= \mathbf{r} \times \mathbf{F} = (2\mathbf{i} + 4\mathbf{j}) \times 600(\mathbf{i} \cos 40^\circ - \mathbf{j} \sin 40^\circ) \\ &= -2610\mathbf{k} \text{ N}\cdot\text{m} \end{aligned}$$

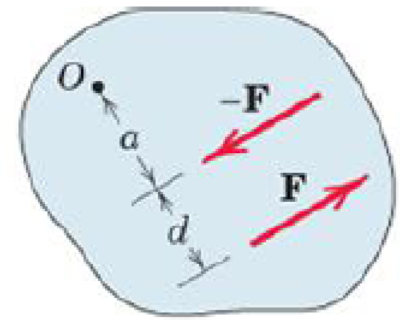
The minus sign indicates that the vector is in the negative z-direction. The magnitude of the vector expression is

$$M_O = 2610 \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$



Moment of a Couple

Moment produced by two equal, opposite and noncollinear forces is called a ***couple***.

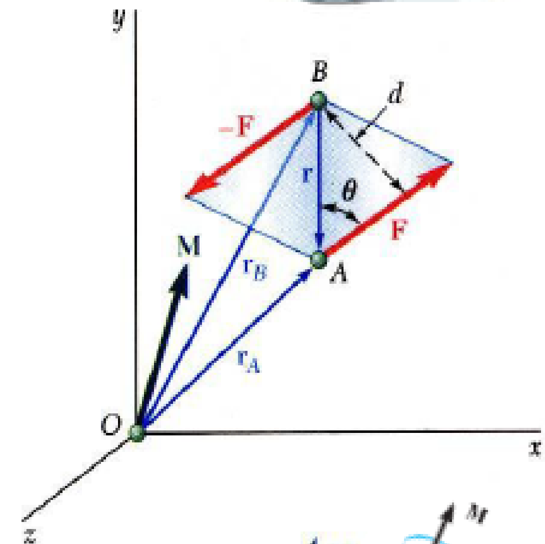


Magnitude of the combined moment of the two forces about O:

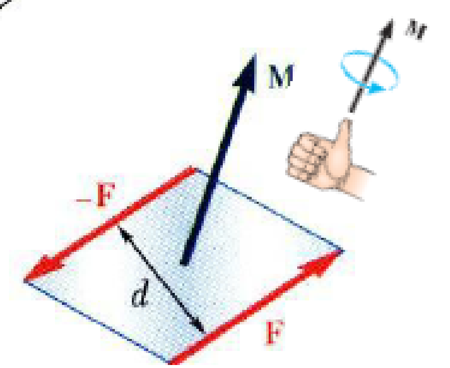
$M = F(a+d) - Fa = Fd$

Vector Algebra Method:
Moment of the couple about point O:

$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right)$$
$$= \left(\vec{r}_A - \vec{r}_B\right) \times \vec{F}$$
$$= \vec{r} \times \vec{F}$$
$$M = rF\sin\theta = Fd$$



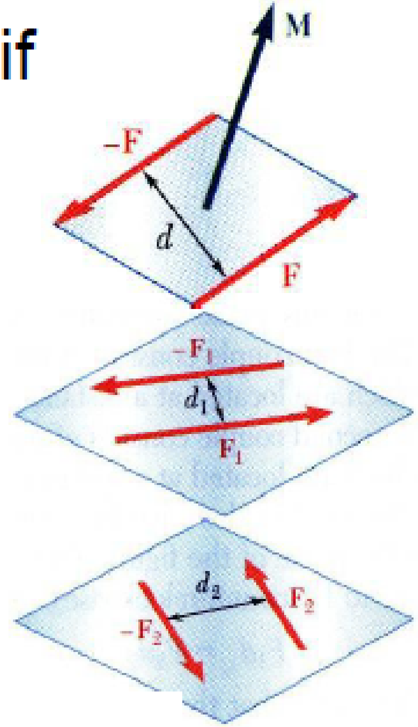
The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a *free vector* that can be applied at any point with the same effect.



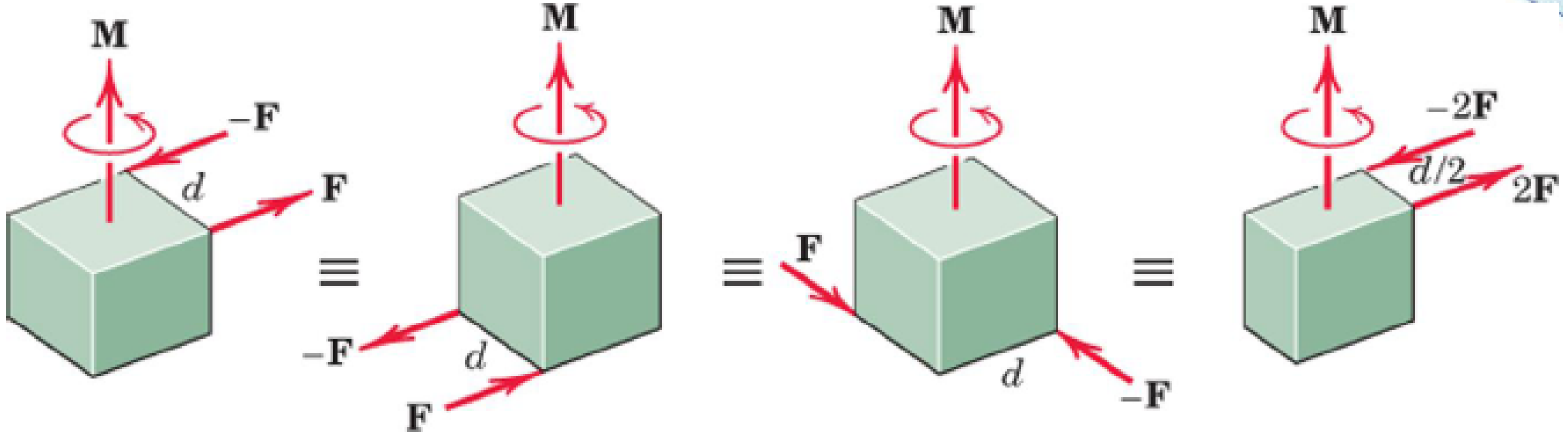
Moment of a Couple

Two couples will have **equal moments** if

- $F_1 d_1 = F_2 d_2$
- the two couples lie in parallel planes, and
- the two couples have the same sense or the tendency to cause rotation in the same direction.



Examples:



Addition of Couples

- Consider two intersecting planes P_1 and P_2 with each containing a couple

$$\vec{M}_1 = \vec{r} \times \vec{F}_1 \text{ in plane } P_1$$

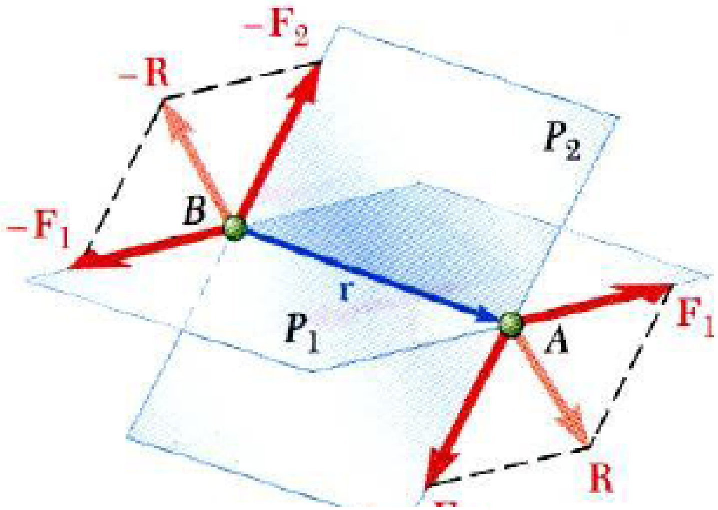
$$\vec{M}_2 = \vec{r} \times \vec{F}_2 \text{ in plane } P_2$$

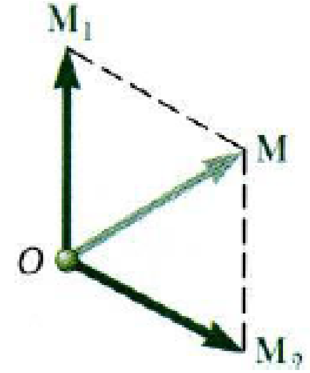
- Resultants of the vectors also form a couple

$$\vec{M} = \vec{r} \times \vec{R} = \vec{r} \times \left(\vec{F}_1 + \vec{F}_2\right)$$

- By Varignon's theorem

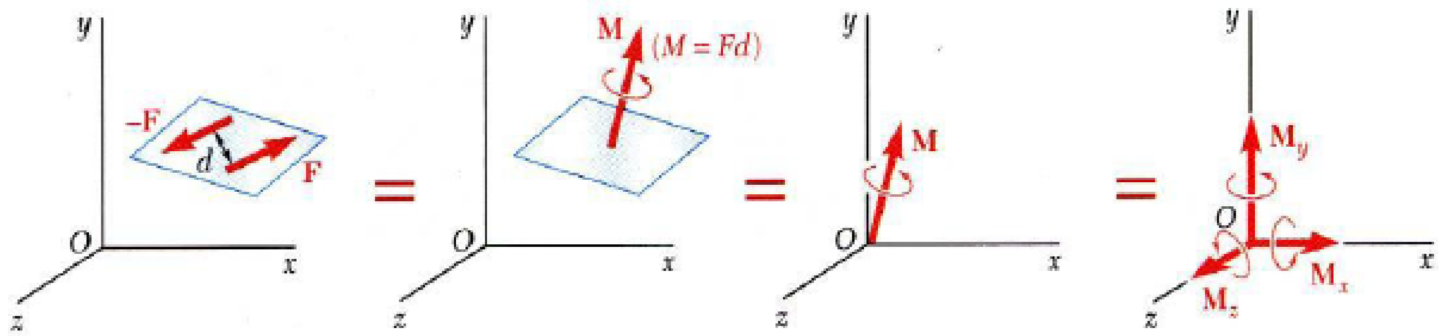
$$\vec{M} = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2$$
$$= \vec{M}_1 + \vec{M}_2$$





- **Sum** of **two couples** is also a couple that is equal to the **vector sum of the two couples**

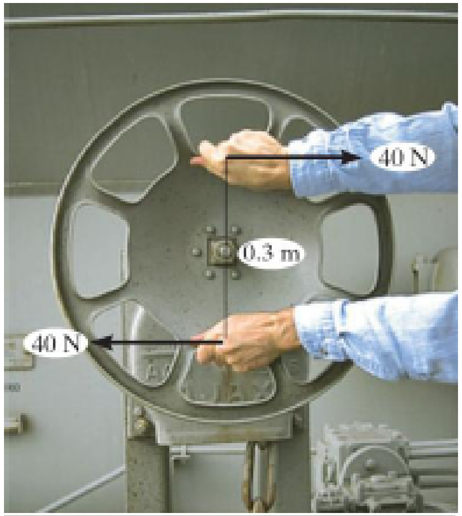
Representation of Couples by Vectors

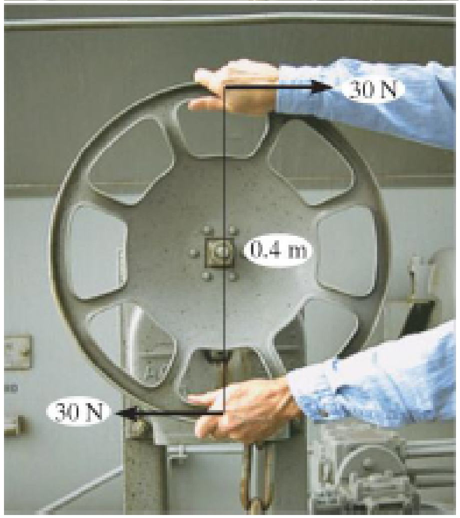


- A couple can be represented by a vector with magnitude and direction equal to the moment of the couple.
- *Couple vectors* obey the law of addition of vectors.
- Couple vectors are **free vectors**, i.e., the point of application is not significant.
- Couple vectors may be resolved into component vectors.

Couple: Example

Moment reqd to turn the shaft connected at center of the wheel = 12 Nm





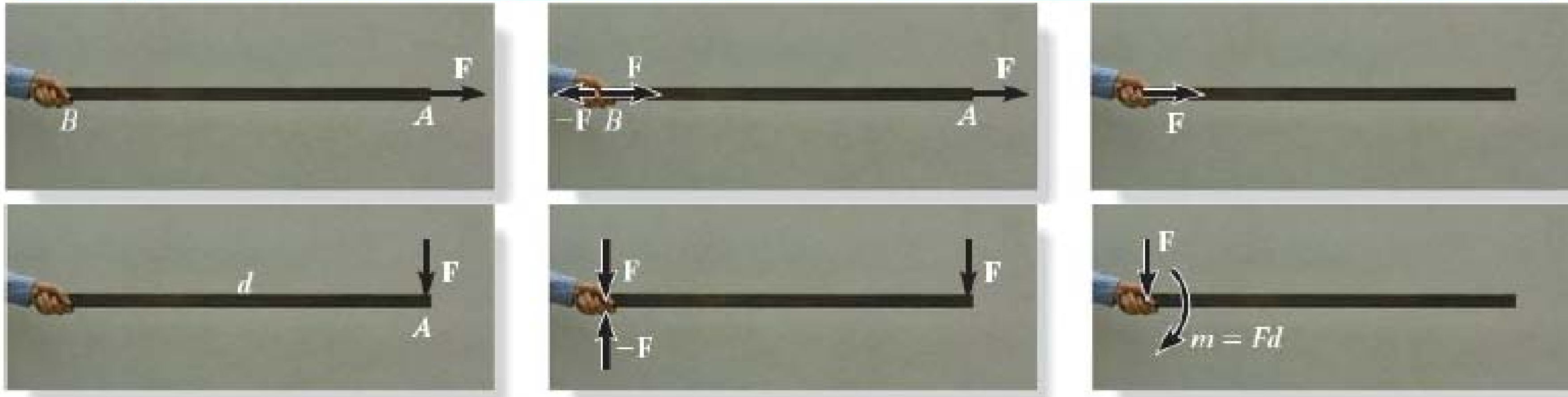
- First case: Couple Moment produced by 40 N forces = 12 Nm
- Second case: Couple Moment produced by 30 N forces = 12 Nm

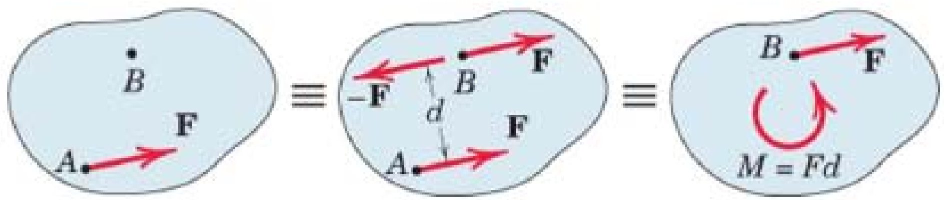
  If only One hand is used → F = 60N

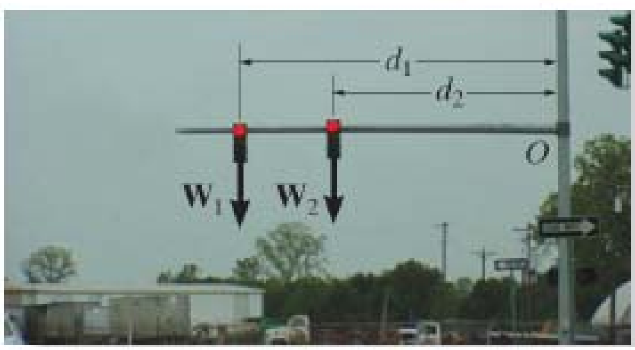
Same couple moment will be produced even if the shaft is not connected at the center of the wheel

→ Couple Moment is a Free Vector

Equivalent Systems (Force-Couple Systems)





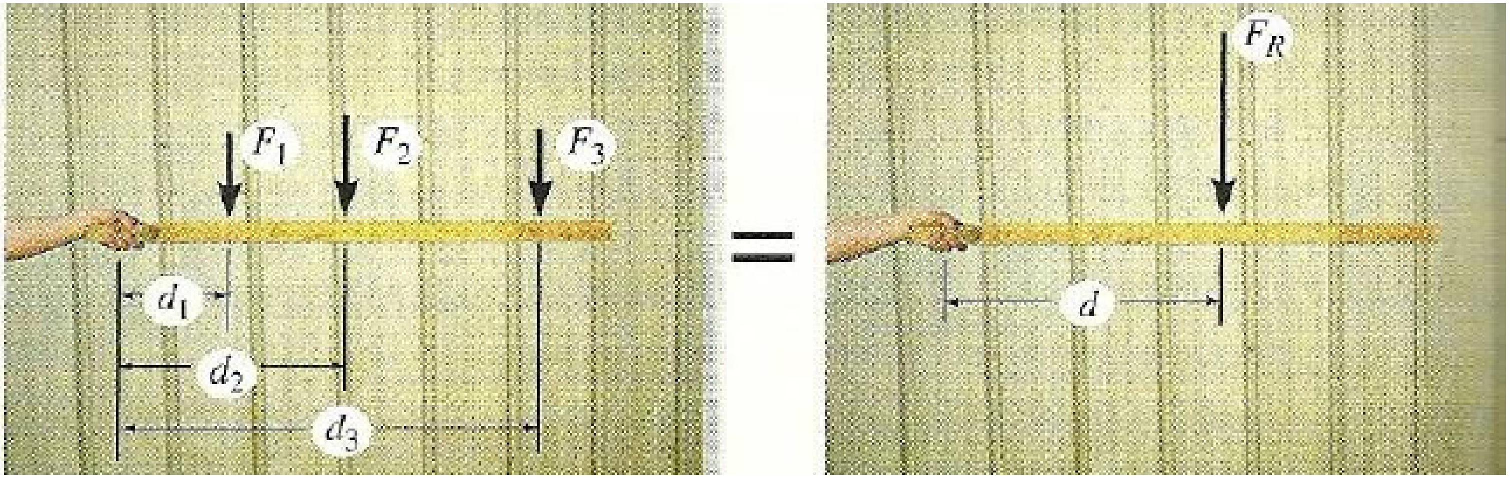


At support O:

$W_R = W_1 + W_2$

$(M_R)_O = W_1 d_1 + W_2 d_2$

Equivalent Systems: Resultants



$F_R = F_1 + F_2 + F_3$

How to find d?

Moment of the Resultant force about the grip must be equal to the moment of the forces about the grip

$F_R d = F_1 d_1 + F_2 d_2 + F_3 d_3$

→ Equilibrium Conditions