١٠٤ ريض

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الساعات المكتبية:

- الأحد – الثلاثاء – الخميس
  من الساعة ٦ صباحا الى الساعة ٨ صباحا.
- الاربعاء
  من الساعة ٦ صباحا الى الساعة ١٠ صباحا.

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تأليف : د. مساعد العبداللطيف – د. مسعود يونخل

مواضيع المقرر:

- القطوع المخروطية:
  - ✓ القطع المكافئ
  - ✓ القطع الناقص
  - ✓ القطع الزائد
- المصفوفات و المحددات:
  - ✓ المصفوفات
  - ✓ المحددات
- أنظمة المعادلات الخطية:
  - ✓ طريقة كرامر
  - ✓ طريقة جاوس
  - ✓ طريقة جاوس جوردان
- طرق التكامل:
  - ✓ التكامل بالتعويض
  - ✓ التكامل بالتجزيء
  - ✓ تكامل الدوال الكسرية
- تطبيقات حساب التكامل:
  - ✓ حساب المساحات
  - ✓ حساب حجوم الأجسام الدورانية
    - ☒ طريقة الأقراص الدائرية
    - ☒ طريقة الشرائح الاسطوانية
- التفاضل الجزئي:
  - ✓ دوال في أكثر من متغير
  - ✓ قاعدة السلسة
  - ✓ حساب مشتقة الدالة الضمنية
- المعادلات التفاضلية
- الاحداثيات القطبية

التقييم :

- الإختبار الفصلي الأول: ٢٥ درجة   التاريخ   ٠٠ -٠٠- ١٤٣٨هـ الوقت: ٧ - ٨:٣٠ مساء
- الإختبار الفصلي الثاني: ٢٥ درجة   التاريخ   ٠٠ -٠٠- ١٤٣٨هـ الوقت: ٧ - ٨:٣٠ مساء

التمارين: ١٠ درجات
الإختبار النهائي: ٤٠ درجة
المجموع: ١٠٠ درجة

توزيع درجة التمارين:

اختبارات قصيرة:
عددها: ٢
موعدها: في محاضرة التمارين في الاسبوع الذي يسبق الاختبار الفصلي.
درجة كل اختبار: ٥ درجات.

الحضور و الحرمان:

- على الطلاب حضور جميع المحاضرات.
- في حالة الغياب: يحق للطالب أن يتغيب عن ٢٥% من المحاضرات أي ما يعادل ١٢ محاضرة فقط.

التغيب عن الاختبار بعذر:

في حالة حصول ظرف على الطالب يمنعه من دخول الاختبار، يرجى تقديم طلب اختبار بديل في اليوم التالي من الاختبار و ذلك عن طريق سكرتارية قسم الرياضيات.

الواجب:

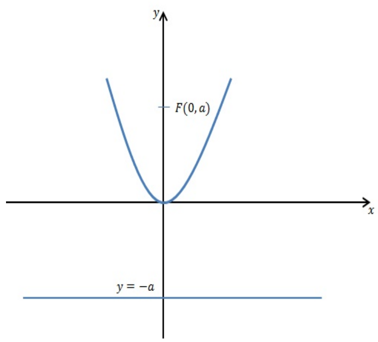
١. جميع التمارين.
٢. تسلم الواجبات في الوقت المحدد.
٣. يجب أن تكتب معلوماتك كاملة: **الاسم – الرقم الجامعي – المقرر .**
٤. لن تقبل الواجبات المتأخرة الا بعذر يسجل في أول ورقة في الواجب

(1) Parabola

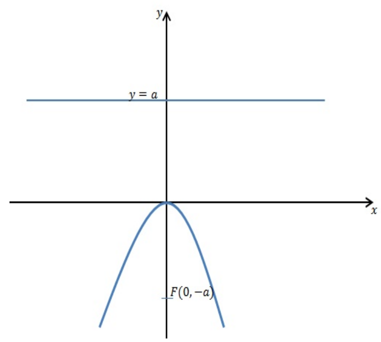
> **Definition 1** *A parabola is the set of all points in the plane equidistant from a fixed point F (called the focus) and a fixed line D (called the directrix) in the same plane.*

(1) The vertex of the parabola is the origin $(0,0)$:

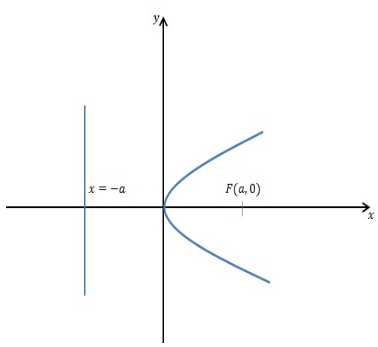
(A) $x^2 = 4ay$, where $a > 0$
- Vertex: $V(0,0)$
- It opens upwards
- Its axis: the y-axis
- Focus: is $F(0,a)$
- Directrix: $y = -a$



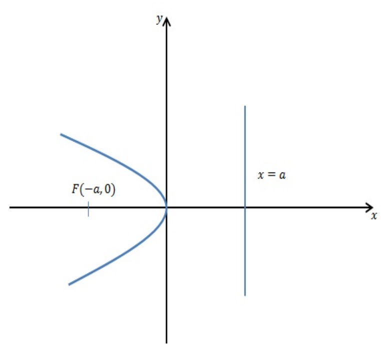
(B) $x^2 = -4ay$, where $a > 0$
- Vertex: $V(0,0)$
- It opens downwards
- Its axis: y-axis
- Focus: $F(0,-a)$
- Directrix: $y = a$



(C) $y^2 = 4ax$, where $a > 0$
- Vertex: $V(0,0)$
- It opens to the right
- Its axis: the x-axis
- Focus is $F(a,0)$
- Directrix: $x = -a$



(D) $y^2 = -4ax$, where $a > 0$
- Vertex: $V(0,0)$
- It opens to the left
- Its axis: x-axis
- Focus: is $F(-a,0)$
- Directrix: $x = a$



Example 1 *Find the focus and the directrix of the parabola $x^2 = 4y$, and sketch its graph.*

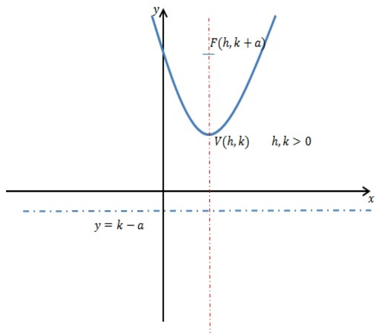
Example 2 *Find the focus and the directrix of the parabola $y^2 = -8x$, and sketch its graph.*

Example 3 *Find the equation of the parabola with focus $(3,0)$ and directrix $x = -3$. Then, sketch the graph.*

(2) The general formula of a parabola:

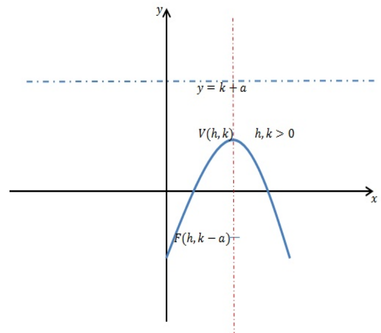
(A) $(x-h)^2 = 4a(y-k)$, where $a > 0$

- Vertex: $V(h,k)$
- It opens upwards
- Its axis: parallel to the y-axis
- Focus: $F(h, k+a)$
- Directrix: $y = k - a$



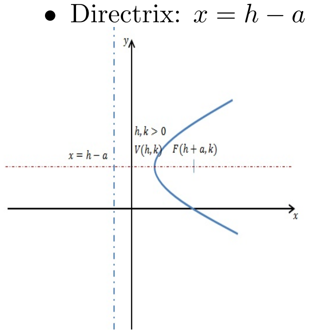
(B) $(x-h)^2 = -4a(y-k)$, where $a > 0$

- Vertex: $V(h,k)$
- Its axis: parallel to the y-axis
- It opens downwards
- Focus: $F(h, k-a)$
- Directrix: $y = k + a$



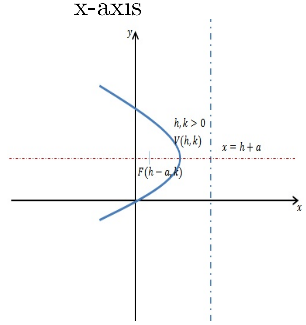
(C) $(y-k)^2 = 4a(x-h)$, where $a > 0$

- Vertex: $V(h,k)$
- Its axis: parallel to the x-axis
- It opens to the right
- Focus: $F(h+a, k)$
- Directrix: $x = h - a$



(D) $(y-k)^2 = -4a(x-h)$, where $a > 0$

- Vertex: $V(h,k)$
- It opens to the left
- Focus: $F(h-a, k)$
- Directrix: $x = h + a$
- Its axis: parallel to the x-axis



Example 4 *Find the focus and the directrix of the parabola* $(x+1)^2 = -4(y-1)$ *, and sketch its graph.*

Example 5 *Find the equation of the parabola with vertex* $(2,1)$ *and focus* $F(2,3)$*. Then, sketch the graph.*

Example 6 *Find the focus and the directrix of the parabola* $2y^2 - 4y + 8x + 10 = 0$ *, and sketch its graph.*

(2) Ellipse

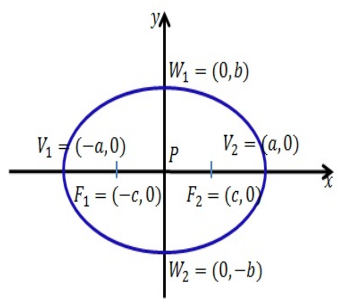
Definition 2 *An ellipse is the set of all points in the plane for which the sum of the distances to two fixed points is constant.*

(1) The center of the ellipse is the origin $(0,0)$:

(A) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **where** $a>b$

$$c=\sqrt{a^2-b^2}$$

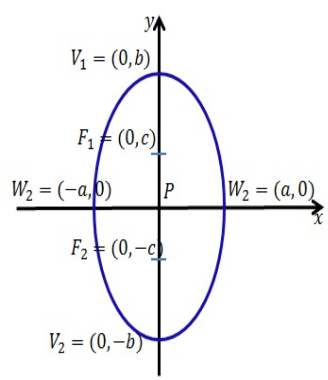
- Center: $P(0,0)$
- Foci: $F_1(-c,0)$, $F_2(c,0)$.
- Vertices: $V_1(-a,0)$, $V_2(a,0)$.
- Major axis: the x-axis, length is $2a$.
- Minor axis: y-axis, length is $2b$.
- Minor axis endpoints: $W_1(0,b)$, $W_2(0,-b)$.



(B) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **where** $a<b$

$$c=\sqrt{b^2-a^2}$$

- Center: $P(0,0)$
- Foci: $F_1(0,c)$, $F_2(0,-c)$
- Vertices: $V_1(0,b)$, $V_2(0,-b)$
- Major axis: y-axis, length is $2b$.
- Minor axis: x-axis, length is $2a$.
- Minor axis endpoints: $W_1(-a,0)$, $W_2(a,0)$.



Example 7 *Identify the features of the ellipse and sketch its graph.*
(a) $9x^2+25y^2=225$ *(b)* $16x^2+9y^2=144$

Example 8 *Find an equation of an ellipse if the center is at the origin and*

(a)
Major axis on x-axis
Major axis length = 14
Minor axis length = 10

(b)
Major axis on y-axis
Minor axis length = 14
Distance of foci from center $=10\sqrt{2}$

==

(2) The general formula of the ellipse:

(A) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ **where** $a > b$, $c = \sqrt{a^2 - b^2}$

- Center: $P(h, k)$
- Foci: $F_1(h-c, k)$, $F_2(h+c, k)$
- Vertices: $V_1(h-a, k)$, $V_2(h+a, k)$
- Major axis: parallel to the x-axis, length is $2a$
- Minor axis: parallel to the y-axis, length is $2b$
- Minor axis endpoints: $W_1(h, k+b)$, $W_2(h, k-b)$

(B) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ **where** $a < b$, $c = \sqrt{b^2 - a^2}$

- Center: $P(h, k)$
- Foci: $F_1(h, k+c)$, $F_2(h, k-c)$
- Vertices: $V_1(h, k+b)$, $V_2(h, k-b)$
- Major axis: parallel to the y-axis, length is $2b$
- Minor axis: parallel to the x-axis, length is $2a$
- Minor axis endpoints: $W_1(h-a, k)$ and $W_2(h+a, k)$

Example 9 *Find the equation of the ellipse with foci at* $(-3, 1)$, $(5, 1)$ *, and one of its vertice is* $(7, 1)$ *, and sketch its graph.*

Example 10 *Find the equation of the ellipse with foci at* $(2, 5)$, $(2, -3)$ *, and the length of its minor axis equals* 6 *, and sketch its graph.*

Example 11 *Identify the features of the ellipse* $4x^2+2y^2-8x-8y-20 = 0$*, and sketch its graph.*

(3) Hyperbola

Definition 3 *A hyperbola is the set of all points in the plane for which the difference of the distances between two fixed points is constant.*

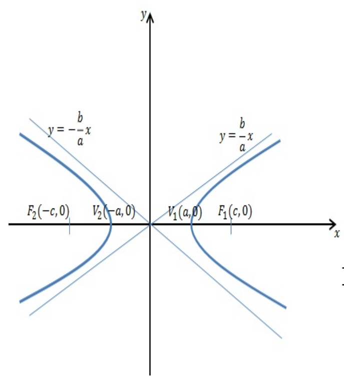
(1) The center of the hyperbola is the origin $(0,0)$:

$$c = \sqrt{a^2 + b^2}$$

The line segment between V_1 and V_2 is the transverse axis.

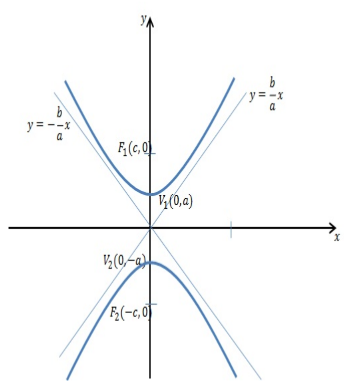
(A) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

- Center: $P(0,0)$
- Foci: $F_1(-c,0)$, $F_2(c,0)$
- Vertices: $V_1(-a,0)$, $V_2(a,0)$
- Transverse axis: x-axis, length is $2a$.
- Asymptotes: $y = \pm\frac{b}{a}x$.



(B) $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

- Center: $P(0,0)$
- Foci: $F_1(0,c)$, $F_2(0,-c)$
- Vertices: $V_1(0,b)$, $V_2(0,-b)$
- Transverse axis: y-axis, length is $2b$
- Asymptotes: $y = \pm\frac{b}{a}x$



Example 12 *Identify the features of the hyperbola and sketch its graph.*
(a) $4x^2 - 16y^2 = 64$ *(b)* $4y^2 - 9x^2 = 36$

Example 13 *Find an equation of a hyperbola if its vertices are* $V_1(3,0)$ *and* $V_2(-3,0)$ *and one of its foci* $(4,0)$.

(2) The general formula of the hyperbola:

$$c = \sqrt{a^2 + b^2}$$

(A) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

- Center: $P(h, k)$
- Foci: $F_1(h-c, k)$, $F_2(h+c, k)$
- Vertices: $V_1(h-a, k)$, $V_2(h+a, k)$
- Transverse axis: parallel to x-axis, length is $2a$
- Asymptotes: $(y-k) = \pm\frac{b}{a}(x-h)$

(B) $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$

- Center: $P(h, k)$
- Foci: $F_1(h, k+c)$, $F_2(h, k-c)$
- Vertices: $V_1(h, k+b)$, $V_2(h, k-b)$
- Transverse axis: parallel to y-axis, length is $2b$
- Asymptotes: $(y-k) = \pm\frac{b}{a}(x-h)$

Example 14 *Find the equation of the hyperbola with foci at* $(-2, 2)$, $(6, 2)$ *and one of its vertices is* $(5, 2)$, *and sketch its graph.*

Example 15 *Find the equation of the hyperbola with foci at* $(-1, -6)$, $(-1, 4)$ *and the length of its transverse axis is* 8 , *and sketch its graph.*

Example 16 *Identify the features of the hyperbola* $2y^2 - 4x^2 - 4y - 8x - 34 = 0$, *and sketch its graph.*

Homework (1):

1. Find the focus and the directrix of the parabola $(x-1)^2 = 8(y+1)$, and sketch its graph.

2. Find the equation of the parabola with vertex $(-4, 2)$ and focus $(-\frac{7}{2}, 2)$. Then, sketch the graph.

3. Find the equation of the parabola with focus $(3, 6)$ and directrix $y = 2$. Then, sketch the graph.

4. Find an equation of an ellipse if the center is at the origin and major axis is on x-axis and its length equals 8 and minor axis length equals 6.

5. Find the equation of the ellipse with foci at $(10, -2), (4, -2)$, and one of its vertices is $(12, -2)$, then sketch its graph.

6. Find an equation of a hyperbola if its vertices are $(0, -2)$ and $(0, 2)$ and one of its foci $(0, \sqrt{13})$.

7. Find the equation of the hyperbola with foci at $(4, -2), (10, -2)$ and one of its vertices is $(8, -2)$, and sketch its graph.

8. Find the elements of the conic section $x^2 + 5y^2 + 6x - 40y + 84 = 0$, and sketch its graph.

9. Find the elements of the conic section $x^2 + 2y + 2x = 2$, and sketch its graph.

10. Find the elements of the conic section $y^2 - 5x^2 + 6y - 40x - 76 = 0$, and sketch its graph.

(1) Matrices

Definition 4 *A matrix A of order $m \times n$ is a set of real numbers arranged in a rectangular array of m rows and n columns.*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Notes :

1. a_{ij} represents the element of the matrix A that lies in row i and column j.
2. The matrix A can also be written as $A = (a_{ij})_{m\times n}$.
3. If the number of rows equals the number of columns ($m = n$), then A is called a square matrix of order n.
4. In a square matrix $A = (a_{ij})$, the set of elements of the form a_{ii} is called the diagonal of the matrix.

Example 17 *Find the order of each matrix, then find the elements:*

1. $A = \begin{pmatrix} 2 & -4 \\ 1 & 0 \end{pmatrix}$, a_{11} *and* a_{22}

2. $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 0 \end{pmatrix}$, a_{12}, a_{21} *and* a_{23}

Example 18 *Find the diagonal of the square matrix.*

1. $A = \begin{pmatrix} 2 & -4 \\ 1 & 0 \end{pmatrix}$

Special types of matrices:

1. **Row vector :** A row vector of order n is a matrix of order $1 \times n$, and it is written as $(a_1\ a_2\ \dots\ a_n)$

Example : $(2\ 7\ 0\ -1\ 9)$ is a row vector of order 5.

2. **Column vector :** A column vector of order n is a matrix of order $n \times 1$, and it is written as $A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

Example: $\begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$ is a column vector of order 3.

3. **Null matrix:** The matrix $(a_{ij})_{m\times n}$ of order $m \times n$ is called a null matrix if $a_{ij} = 0$ for all i and j, and it is denoted by 0.

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Example: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a null matrix of order 2×3.

4. **Upper triangular matrix :** The square matrix $A = (a_{ij})$ of order n is called an upper triangular matrix if $a_{ij} = 0$ for all $i > j$, and it is written as $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$

Example: $\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & 5 \end{pmatrix}$ is an upper triangular matrix of order 3.

5. **Lower triangular matrix :** The square matrix $A = (a_{ij})$ of order n is called a lower triangular matrix if $a_{ij} = 0$ for all $i < j$, and it is written as

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Example : $\begin{pmatrix} 4 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 3 & 5 \end{pmatrix}$ is a lower triangular matrix of order 3.

6. **Diagonal matrix:** The square matrix $A = (a_{ij})$ of order n is called a diagonal matrix if $a_{ij} = 0$ for all ij , and it is written as

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

Example : $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ is a diagonal matrix of order 3.

7. **Identity matrix:** The square matrix $I_n = (a_{ij})$ of order n is called an identity matrix if $a_{ij} = 0$ for all $i = j$ and $a_{ij} = 1$ for all $i = j$, and it is

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Example : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an identity matrix of order 3.

Matrix Operations:

(1) Addition and subtraction of matrices :

Addition or subtraction of two matrices is defined if the two matrices have the same order. If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ any two matrices of order $m \times n$ then

1. $A + B = (a_{ij} + b_{ij})_{m\times n}$.

$$A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

2. $A - B = (a_{ij} - b_{ij})_{m\times n}$.

$$A - B = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \cdots & a_{2n}-b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \cdots & a_{mn}-b_{mn} \end{pmatrix}$$

Example 19 *If* $A = \begin{pmatrix} 1 & 3 & 2 \\ 5 & -4 & 6 \\ 0 & 9 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 5 & 0 & 8 \\ 1 & 4 & -1 \\ 10 & 11 & -2 \end{pmatrix}$

Find $A + B$ *and* $A - B$.

Notes:

1. The addition of matrices is commutative : if A and B any two matrices of the same order then $A + B = B + A$.
2. The null matrix is the identity element of addition : if A is any matrix then $A + 0 = A$.

(2) Multiplying a matrix by a scalar:

If $A = (a_{ij})$ is a matrix of order $m \times n$ and $c \in R$ then $c\ A = (c\ aij)$.

$$c\ A = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{pmatrix}$$

Example 20 *If* $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 9 & 2 \end{pmatrix}$ *, find* $3A$.

Example 21 *If* $A = \begin{pmatrix} 1 & 6 \\ -2 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 3 \\ 0 & 8 \end{pmatrix}$ *, find* $-2A + 3B$.

(3) Multiplying a row vector by a column vector:

If $A = (a_1\ a_2\ ...\ a_n)$ is a row vector of order n and B $= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is a column vector of order n then $AB = (a_1\ a_2\ ...\ a_n)B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} =$ $a_1b_1 + a_2b_2 + ... + a_nb_n$

Example 22 *If* $A = (2\ 1\ 4)$ *and* $B = \begin{pmatrix} 8 \\ -3 \\ 5 \end{pmatrix}$*, find* AB

Example 23 *If* $A = (2\ 1\ 4\ \ -6)$ *and* $B = \begin{pmatrix} 1 \\ 3 \\ -4 \\ 7 \end{pmatrix}$*, find* AB

(4) Multiplication of matrices:

1. If A and B any two matrices then AB is defined if the number of columns of A equals the number of rows of B.

2. If $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{n \times}$ then $AB = (c_{ij})_{m \times p}$.
c_{ij} is calculated by multiplying the i^{th} row of A by the j^{th} column of B.

Example 24 *If* $A = \begin{pmatrix} 1 & 6 \\ -2 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 3 \\ 0 & 8 \end{pmatrix}$ *, find* AB.

Example 25 *If* $A = \begin{pmatrix} 1 & 6 & 2 \\ -2 & 4 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 4 & 2 \\ 0 & 1 & 7 \end{pmatrix}$ *, find* AB.

Example 26 *If* $A = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix}$ *,* $B = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ *and* $C = \begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 1 \end{pmatrix}$ *,*

find 1. AB *2.* BC.

Notes: $\boxed{1}$ **Matrix multiplication is not commutative.**

Example 27 *If* $A = \begin{pmatrix} 4 & 3 & 9 \\ -1 & 2 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ *Compute (if possible) 1.* AB *2.* BA

Example 28 *If* $A = \begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 7 & 5 \\ 1 & 0 \end{pmatrix}$ *, find 1.* AB *2.* BA.

$\boxed{2}$ **The identity matrix is the identity element in matrix multiplication.**

$$\mathrm{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & \end{pmatrix}$$

If A is a matrix of order $m \times n$ and I_n is the identity matrix of order n then

$$AI_n = I_nA = A\ .$$

Example 29 *If* $A = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}$ *, find* AI_2

Properties of operations on matrices:

1. If A, B and C any three matrices of the same order then
$$A + B + C = (A + B) + C = A + (B + C) = (A + C) + B$$
2. If A, B any two matrices of order $m \times n$ and C a matrix of order $n \times p$ then
$$(A + B)C = AC + BC$$
3. If A, B any two matrices of order $m \times n$ and C a matrix of order $p \times m$ then
$$C(A + B) = CA + CB$$
4. If A a matrix of order $m \times n$, B a matrix of order $n \times p$ and C a matrix of order $p \times q$ then
$$ABC = (AB)C = A(BC)$$

Transpose of a matrix :

If $A = (a_{ij})_{m\times n}$ then the transpose of A is $A^t = (a_{ji})_{n\times m}$.

Example 30 *If* $A = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}$ *, find* A^t

Note : The transpose of a lower triangular matrix is an upper triangular matrix, and the transpose of an upper triangular matrix is a lower triangular matrix .

Properties of transpose of a matrix :

If A and B any two matrices and $\lambda \in R$ then
1. $(A^t)^t = A$.
2. $(A + B)^t = A^t + B^t$.
3. $(\lambda A)^t = \lambda A^t$.
4. $(AB)^t = B^t A^t$.

===

(2) Determinants

If A is a **square** matrix then the determinant of A is denoted by $det(A)$ or $|A|$.

(A) The determinant of a 2×2 matrix :

If A $= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

Example 31 *find* $det(A)$*: (1)* $A = \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix}$ *(2)* $A = \begin{pmatrix} 4 & -1 \\ 2 & 9 \end{pmatrix}$

(B) The determinant of a 3×3 matrix:

If A $= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then

$$det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example 32 *Find* $|A|$*: (1)* $A = \begin{pmatrix} 1 & 6 & 3 \\ 5 & -1 & 4 \\ -2 & 9 & 7 \end{pmatrix}$ *(2)* $A = \begin{pmatrix} 4 & 1 & 5 \\ 2 & 1 & -2 \\ 1 & 8 & 7 \end{pmatrix}$

(C) The determinant of a 4×4 matrix :

If A $= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$, then

$$det(A) = a_{11}det(A_1) - a_{12}det(A_2) + a_{13}det(A_3) - a_{14}det(A_4)$$

where $A_1 = \begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}$ $A_2 = \begin{pmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$

$A_3 = \begin{pmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix}$ $A_4 = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$

Example 33 *Find* $det(A)$*:*

(1) $A = \begin{pmatrix} 1 & 6 & 3 & 2 \\ 5 & -1 & 4 & 1 \\ -2 & 9 & 7 & 3 \\ 7 & 1 & 3 & -6 \end{pmatrix}$ *(2)* $A = \begin{pmatrix} 4 & 1 & 5 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 3 & 9 \\ 1 & 7 & 4 & 6 \end{pmatrix}$

==

Properties of determinants:

1. If A is a square matrix that contains a zero row (or a zero column) then $det(A) = 0$.

2. If A is a square matrix that contains two equal rows (or two equal columns) then $det(A) = 0$.

3. If A is a square matrix that contains a row which is a multiple of another row (or a column which is a multiple of another column) then $det(A) = 0$.

4. If A is a diagonal matrix or an upper triangular matrix or a lower triangular matrix the $det(A)$ is the the product of the elements of the main diagonal.

5. The determinant of the null matrix is 0 and the determinant of the identity matrix is 1.

6. If A is a square matrix and B is the matrix formed by multiplying one of the rows (or columns) of A by a non-zero constant λ then $det(B) = \lambda det(A)$.

7. If A is a square matrix and B is the matrix formed by interchanging two rows (or two columns) of A then $det(B) = -det(A)$.

8. If A is a square matrix and B is the matrix formed by multiplying a row by a non-zero constant and adding the result to another row (or multiplying a column by a non-zero constant and adding the result to another column) then $det(B) = det(A)$.

Example 34 *Use properties of determinants to calculate the determinants of the following matrices*

(1) $A = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 3 & 4 & 7 \end{pmatrix}$ *(2)* $A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & 5 & 6 \\ 3 & 4 & 3 \end{pmatrix}$

(3) $A = \begin{pmatrix} 1 & 2 & -2 \\ 4 & 7 & 5 \\ 3 & 6 & -6 \end{pmatrix}$ *(4)* $A = \begin{pmatrix} 3 & 0 & 4 \\ 6 & -1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$

(5) $A = \begin{pmatrix} 5 & 2 & 3 \\ 15 & 8 & 1 \\ 10 & 6 & 2 \end{pmatrix}$ *(6)* $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{pmatrix}$

==

Homework (2):

Question 1: If A = $\begin{pmatrix} 1 & 3 & 2 \\ 5 & -4 & 6 \\ 0 & 9 & 2 \end{pmatrix}$, B = $\begin{pmatrix} 5 & 0 \\ 1 & 4 \\ 10 & 11 \end{pmatrix}$ and C = $\begin{pmatrix} -2 & 0 \\ 0 & 7 \\ 5 & 3 \end{pmatrix}$

Find

1. $B + C$
2. $2B + 3C$
3. $C - B$
4. $A - C$
5. AB
6. BA
7. A^t
8. $(3A)^t$
9. $det(A)$
10. $det(2A)$

Question 2: If $B = 7A$ and $det(A) = -3$, find $det(B)$.

Question 3: Find the following determinants:

1. $\begin{vmatrix} 1 & -2 \\ 2 & 7 \end{vmatrix}$
2. $\begin{vmatrix} 1 & -2 & 3 \\ 4 & 0 & 1 \\ 2 & 7 & 0 \end{vmatrix}$

==

Consider the system of linear equations in 2 different variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

The above system of linear equations can be written as : $AX = B$

where A = $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, X = $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and B = $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Generally, consider the system of linear equations in n different variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ... + a_{3n}x_n &= b_3 \\ \vdots + \vdots + \vdots + \vdots + \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ... + + a_{nn}x_n &= b_n \end{aligned} \tag{1}$$

The above system of linear equations can be written as : $AX = B$

where A = $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, X = $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, and B = $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

A is called the coefficients matrix
X is called the column vector of variables (or column vector of the unknowns)
B is called the column vector of constants (or column vector of the resultants)

Theorem 1
The system of linear equations (3) has a solution if $det(A) \neq 0$.

This chapter presents three methods of solving the system of linear equations (3), the first method is **Cramers rule** , the second is **Gauss elimination method**, and the third is **Gauss-Jordan method**.

(1) Cramers Rule
Consider the system of linear equations in n different variables (3).
The method:
If $det(A) \neq 0$, then the solution of the system (3) is given by

$$x_i = \frac{det(A_i)}{det(A)} \quad \text{for every} i = 1, 2, ..., n \ .$$

where A_i is the matrix formed by replacing the i^{th} column of A by the column vector of constants:

A$_1$ = $\begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, A_2 = $\begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{pmatrix}$, ..., A_n = $\begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{pmatrix}$,

Example 35 *Use Cramers rule to solve the system of linear equations*

$$\begin{aligned} 2x + 3y &= 7 \\ -x + y &= 4 \end{aligned}$$

Example 36 *Use Cramers rule to solve the system of linear equations*

$$\begin{aligned} 2x + y + z &= 3 \\ 4x + y - z &= -2 \\ 2x - 2y + z &= 6 \end{aligned}$$

Example 37 *Use Cramers rule to solve the system of linear equations*

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + x_2 &= -1 \end{aligned}$$

Example 38 *Use Cramers rule to solve the system of linear equations*

$$\begin{aligned} x + y + z &= 12 \\ x - y &= 2 \\ x - z &= 4 \end{aligned}$$

(2) Gauss elimination method

Consider the system of linear equations in n different variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ... + a_{3n}x_n &= b_3 \\ \vdots + \vdots + \vdots + \vdots + \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ... + +a_{nn}x_n &= b_n \end{aligned} \tag{2}$$

$$AX = B$$

where A = $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, X = $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, and B = $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

The method:

1. Construct the augmented matrix $[A|B]$.

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right)$$

2. Use elementary row operations on the augmented matrix to transform the matrix A to an upper triangular matrix with leading coefficient of each row equals 1.

$$\left(\begin{array}{cccccc|c} 1 & c_{12} & c_{13} & c_{14} & \cdots & c_{1n} & d_1 \\ 0 & 1 & c_{33} & c_{24} & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & c_{(n-1)n} & d_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 1 & d_n \end{array}\right)$$

3. From the last augmented matrix, $x_n = d_n$
and the rest of the unknowns can be calculated by backward substitution.

Example 39 *Use Gauss elimination method to solve the system*

$$\begin{aligned} x - 2y + z &= 4 \\ -x + 2y + z &= -2 \\ 4x - 3y - z &= -4 \end{aligned}$$

Example 40 *Use Gauss elimination method to solve the system*

$$\begin{aligned} x + y + z &= 2 \\ x - y + 2z &= 0 \\ 2x + z &= 2 \end{aligned}$$

Example 41 *Use Gauss elimination method to solve the system*

$$\begin{aligned} 3x_1 + x_2 &= 9 \\ x_1 + 2x_2 &= 8 \end{aligned}$$

Example 42 *Use Gauss elimination method to solve the system*

$$\begin{aligned} x + 2y + 3z &= 14 \\ 2x + y + 2z &= 10 \\ 3x + 4y - 3z &= 2 \end{aligned}$$

(3) Gauss-Jordan method

Consider the system of linear equations in n different variables:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ... + a_{3n}x_n &= b_3 \\
\vdots + \vdots + \vdots + \vdots + \vdots &= \vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ... + +a_{nn}x_n &= b_n
\end{aligned} \tag{3}$$

$$AX = B$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

The method:

1. Construct the augmented matrix $[A|B]$.

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right)$$

2. Use elementary row operations on the augmented matrix to transform the matrix A to the identity matrix .

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & \cdots & 0 & d_1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & d_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 1 & d_n \end{array}\right)$$

3. From the last augmented matrix, $x_i = d_i$ for every $i = 1, 2, , n$.

Example 43 *Use Gauss-Jordan method to solve the system*

$$\begin{aligned} x + y &= 2 \\ 2x + y &= 1 \end{aligned}$$

Example 44 *Use Gauss-Jordan method to solve the system*

$$\begin{aligned} x + y + z &= 2 \\ x - y + 2z &= 0 \\ 2x + z &= 2 \end{aligned}$$

Example 45 *Use Gauss-Jordan method to solve the system*

$$\begin{aligned} x - 2y + 2z &= 5 \\ 5x + 3y + 6z &= 57 \\ x + 2y + 2z &= 21 \end{aligned}$$

==

Homework (3):

Question 1: Use Cramers rule to solve the system of linear equations

1.

$$\begin{aligned} x+y+z&=18\\ x-y+z&=6\\ x+y-z&=4 \end{aligned}$$

2.

$$\begin{aligned} 2x-4y+3z&=10\\ 3x+y-2z&=6\\ x+3y-z&=20 \end{aligned}$$

Question 2: Use Gauss elimination method to solve the system

1.

$$\begin{aligned} x+y+z&=18\\ x-y+z&=6\\ x+y-z&=4 \end{aligned}$$

2.

$$\begin{aligned} x+y+z&=12\\ x-y&=2\\ x-z&=4 \end{aligned}$$

Question 3: Use Gauss-Jordan method to solve the system

1.

$$\begin{aligned} x-3y+z&=21\\ 4x+2y+z&=14\\ 3x+3y+z&=7 \end{aligned}$$

2.

$$\begin{aligned} 2x-4y+3z&=10\\ 3x+y-2z&=6\\ x+3y-z&=20 \end{aligned}$$

Anti-derivatives & Indefinite integral

(1) Anti-derivatives

Anti-derivatives

Definition 5 *A function F is called an anti-derivative of f on an interval I if*

$$F^{'}(x) = f(x) \text{ for every } x \in I \; .$$

Example 46

1. *Let $F(x) = x^2 + 3x + 1$ and $f(x) = 2x + 3$.*
   Since $F^{'}(x) = f(x)$, the function $F(x)$ is an anti-derivative of $f(x)$.

2. *Let $G(x) = \sin(x) + x$ and $g(x) = \cos(x) + 1$.*
   We know that $G^{'}(x) = \cos(x) + 1$ and this means the function $G(x)$ is an anti-derivative of $g(x)$.

Theorem 2 *If the functions $F(x)$ and $G(x)$ are anti-derivatives of a function $f(x)$ on the interval I, there exists a constant c such that $G(x) - F(x) = c$.*

Example 47 *Let $f(x) = 2x$. The functions*
$F(x) = x^2 + 2$,
$G(x) = x^2 - \frac{1}{2}$,
$H(x) = x^2 - \sqrt[3]{2}$,
and many other functions are anti-derivatives of a function $f(x)$. Generally, for the function $f(x) = 2x$, the function $F(x) = x^2 + c$ is the anti-derivative where c is an arbitrary constant.

Example 48 *Find the general form of the anti-derivative of $f(x) = 6x^5$.*

(2) Indefinite Integrals

Indefinite Integrals

Definition 6 *Let f be a continuous function on an interval I. The indefinite integral of $f(x)$ is the general anti-derivative of $f(x)$ on I and symbolized by*

$$\int f(x)\; dx = F(x) + c \; .$$

The function $f(x)$ is called the integrand, the symbol $\int$ is the integral sign, x is called the variable of integration and c is the constant of integration.

(3) Properties of Indefinite Integrals

Let f and g be integrable functions, then

1. $\int \big(f(x) \pm g(x)\big)\; dx = \int f(x) \pm \int g(x)\; dx \; .$

2. $\int kf(x)\; dx = k\int f(x)\; dx$, where k is a constant

Basic Integration Rules:

1) $\int x^n\; dx = \frac{x^{n+1}}{n+1} + c$ where $n \neq -1$.

Special case: $\int 1\; dx = x + c$.

Example 49 *Evaluate the following integrals:*

1. $\int x + 3\; dx$

2. $\int 4x^3 + 2x + 1\; dx$

2) Trigonometric Functions:

$$\int \cos\ dx = \sin\ x + c$$

$$\int \sin\ x\ dx = -\cos\ x + c$$

$$\int \sec^2\ x\ dx = \tan\ x + c$$

$$\int \csc^2\ x\ dx = -\cot\ x + c$$

$$\int \sec\ x\ \tan\ x\ dx = \sec\ x + c$$

$$\int \csc\ x\ \cot\ x\ dx = -\csc\ x + c$$

Example 50 *Evaluate the following integrals:*

1. $\int 2\sin x + 3\cos x\ dx$

2. $\int \sqrt{x} + \sec^2 x\ dx$

3) Natural Logarithmic and Exponential Functions

$$\int \frac{1}{x}\ dx = \ln x + c$$

$$\int e^x\ dx = e^x + c$$

Example 51 *Evaluate the following integrals:*

1. $\int \frac{2}{x} - \csc^2 x\ dx$

2. $\int 3e^x + \frac{1}{1+x^2}\ dx$

3. $\int \frac{1}{x} + \frac{1}{x^2}\ dx$

4) Inverse Trigonometric Functions

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + c \text{ where } |x| < 1$$

$$\int \frac{1}{1+x^2}\ dx = \tan^{-1} x + c$$

$$\int \frac{1}{x\sqrt{x^2-1}}\ dx = sec^{-1}x + c \text{ where } |x| > 1$$

Example 52 *Evaluate the following integrals:*

1. $\int \frac{3}{1+x^2}\ dx$

2. $\int \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x}}\ dx$

(4) Definite Integrals:

Definite Integrals

Definition 7 *Let f be a function defined on a closed interval $[a, b]$. If f is integrable on that interval and $F(x)$ is the general anti-derivative, the definite integral of f is*

$$\int_a^b f(x)\ dx = \Big[F(x)\Big]_a^b = F(b) - F(a)\ .$$

The numbers a and b are called the limits of the integration.

Example 53 *Evaluate the following integrals:*

1. $\displaystyle\int_{-1}^{2} 2x + 1\ dx$

2. $\displaystyle\int_{0}^{3} x^2 + 1\ dx$

3. $\displaystyle\int_{1}^{2} \frac{1}{\sqrt{x^3}}\ dx$

4. $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin(x) + 1\ dx$

5. $\displaystyle\int_{\frac{\pi}{4}}^{\pi} \sec^2(x) - 4\ dx$

6. $\displaystyle\int_{0}^{1} 2x + e^x\ dx$

7. $\displaystyle\int_{2}^{1} \frac{1}{x} + \sqrt{x}\ dx$

(5) Integration by Substitution:

Substitution Method

Theorem 3 *Let g be a differentiable function on the interval I where the derivative is continuous. Let f be a continuous on an interval I involves the range of the function g. If F is an anti-derivative of the function f on I, then*

$$\int f(g(x))g^{'}(x)\ dx = F(g(x)) + c, \quad x \in I\ .$$

Steps of Integration by Substitution:
For simplicity, the substitution method can be summarized in the following steps:
Step 1: Choose a new variable u.
Step 2: Determine the value of du.
Step 3: Make the substitution i.e., eliminate all occurrences of x in the integral by making the entire integral is in terms of u.
Step 4: Evaluate the new integral.
Step 5: Return the evaluation to the initial variable x.

Example 54 *Evaluate the integral* $\int (x+1)^3\ dx$.

Example 55 *Evaluate the integral* $\int 2x(x^2+1)^3\ dx$.

Example 56 *Evaluate the integral* $\int x^2\sqrt{x^3+1}\ dx$.

Example 57 *Evaluate the following integrals:*

1. $\int 3\cos(3x+4)\ dx$
2. $\int x\sec^2(x^2)\ dx$
3. $\int \cos x\ e^{\sin x}\ dx$
4. $\int \dfrac{2x}{x^2+1}\ dx$

(6) Integration by Parts:
Integration by parts is a method to transfer the original integral to an easier integral that can be evaluated. Practically, the integration by parts divides the original integral into two parts u and dv. Then, we try to find the du by deriving u and v by integrating dv.

Integration by Parts

Theorem 4 *If $u = f(x)$ and $v = g(x)$ such that $f'(x)$ and $g'(x)$ are continuous, then*

$$\int u\ dv = uv - \int v\ du\ .$$

Proof. We know that $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$. Thus, $f(x)g'(x) = \frac{d}{dx}(f(x)g(x)) - f'(x)g(x)$.
By integrating the both sides, we have

$$\begin{aligned}\int f(x)g'(x)\ dx &= \int \frac{d}{dx}(f(x)g(x))\ dx - \int f'(x)g(x)\ dx \\ &= f(x)g(x) - \int f'(x)g(x)\ dx\ .\end{aligned}$$

Since $u = f'(x)$ and $dv = g'(x)$, then

$$\int u\ dv = uv - \int v\ du\ . \blacksquare$$

Theorem 4 shows that the integration by parts transfers the integral $\int u\ dv$ into the integral $\int v\ du$ that should be easier than the original integral. The question here is, what we choose as $u(x)$ and what we choose as $dv = v'(x)\ dx$. It is useful to choose u as a function that simplifies when differentiated, and to choose v' as a function that simplifies when integrated. This is explained in a clearer sight through the following examples.

Example 58 *Evaluate the following integral* $\int x\ \cos x\ dx$.

Example 59 *Evaluate the following integral* $\int x\ e^x\ dx$.

Remark 1
1. *Remember that when we consider the integration by parts, we want to have an easier integral. As we saw in Example 59, if we choose $u = e^x$ and $dv = x\ dx$ we have $\int \frac{x^2}{2} e^x\ dx$ which is more difficult than the original one.*

2. *When considering the integration by parts, we have to choose dv a function that can be integrated (see Example 60).*

3. *Sometimes we need to use the integration by parts two times as in Examples 61 and 62.*

Example 60 *Evaluate the following integral* $\int \ln x \, dx$.

Example 61 *Evaluate the following integral* $\int e^x \cos x \, dx$.

Example 62 *Evaluate the following integral* $\int x^2 e^x \, dx$.

Example 63 *Evaluate the following integral* $\int_0^1 \tan^{-1} x \, dx$.

(7) Integrals of Rational Functions:
We study rational functions of the form $q(x) = \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials. The previous techniques like integration by parts is not enough to evaluate the integral of the rational functions. Therefore, we need to know a new technique to help us to evaluate the integral of the rational functions. This technique is called decomposition of the rational functions into a sum of partial fractions.

The practical steps of integrals of rational functions can be summarized as follows:

Step 1: If the degree of $g(x)$ is less than the degree of $f(x)$, we do polynomial long-division, otherwise we move to step 2.
From the long division shown on the right side, we have

$$q(x) = \frac{f(x)}{g(x)} = h(x) + \frac{r(x)}{g(x)} ,$$

where $h(x)$ is called the quotient and $r(x)$ is called the remainder.

$$\begin{array}{r|l} & \text{h(x)} \\ \hline g(x) & \text{f(x)} \\ & \ldots \\ & \ldots \\ \hline & \text{r(x)} \end{array}$$

Step 2: Factor the denominator $g(x)$ into irreducible polynomials where the result is either linear or irreducible quadratic polynomials.

Step 3: Find the partial fraction decomposition. This step depends on step 2 where if degree of $f(x)$ is less than the degree of $g(x)$, then the fraction $\frac{f(x)}{g(x)}$ can be written as a sum of partial fractions:

$$q(x) = P_1(x) + P_2(x) + P_3(x) + \ldots + P_n(x) ,$$

where each $P_i(x) = \frac{A}{(ax+b)^m}, m \in \mathbb{N}$ or $P_i(x) = \frac{Ax+B}{(ax^2+bx+c)^m}$ if $b^2 - 4ac < 0$. The constants $A, B, \ldots$ are computed later.

Step 4: Integrate the result of step 3.

Example 64 *Evaluate the following integral* $\displaystyle\int \frac{x+1}{x^2-2x-8}\, dx$.

Example 65 *Evaluate the integral* $\displaystyle\int \frac{x+3}{(x-3)(x-2)}\, dx$.

Example 66 *Evaluate the integral* $\int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 + 3x + 2}\, dx.$

Remark 2

1. *The number of constants $A, B, C, \ldots$ is equal to the degree of the denominator $g(x)$. Therefore, in the case of repeated factors of the denominator, we have to check the number of the constants and the degree of $g(x)$.*
2. *If the denominator $g(x)$ contains on irreducible quadratic factors, the numerators of fraction decomposition should be polynomials of degree one.*

Example 67 *Evaluate the following integral* $\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)} dx.$

Example 68 *Evaluate the following integral* $\int \frac{2x^2 + 3x + 2}{x^3 + x} dx.$

Homework (4):

Question 1: Evaluate the following integrals:

1. $\displaystyle\int sec^2(3x-5)\ dx$
2. $\displaystyle\int \frac{dx}{\sqrt{16-x^2}}$
3. $\displaystyle\int xe^x\ dx$
4. $\displaystyle\int x\cos x\ dx$
5. $\displaystyle\int \sin^{-1} x\ dx$
6. $\displaystyle\int \frac{dx}{x^2-x-2}$
7. $\displaystyle\int x(2x^2-3)^8\ dx$
8. $\displaystyle\int \frac{\cos\sqrt[3]{x}}{\sqrt[3]{x^2}}\ dx$

Question 2: Evaluate the following integrals:

1. $\displaystyle\int \frac{secx+\tan x}{\cos x}\ dx$
2. $\displaystyle\int x\ln\sqrt{x}\ dx$
3. $\displaystyle\int_1^3 x^2+1\ dx$
4. $\displaystyle\int_e^5 \frac{1}{x-2}\ dx$
5. $\displaystyle\int_3^6 \frac{1}{x-2}+\frac{2}{x+1}\ dx$
6. $\displaystyle\int_0^{\pi/2} (1+\sqrt{\cos x})^2\sin x\ dx$

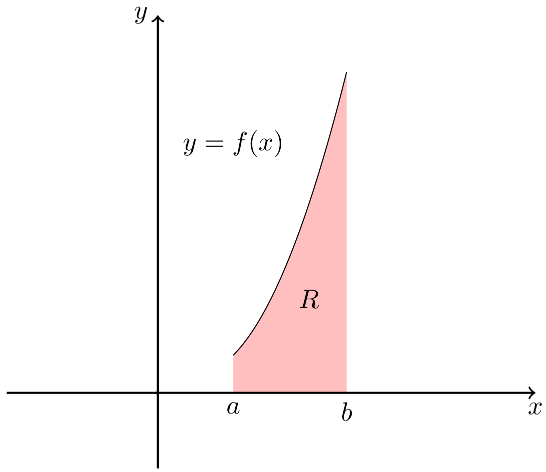
(1) Area

Integration can be used to calculate areas under bounded graphs. In general, we consider the following cases:

1. the area between a curve, the x-axis or y-axis, and two given ordinates,
2. the area between a curve, the x-axis or y-axis, and two ordinates given by crossing the curve the axis,
3. the area between two curves.

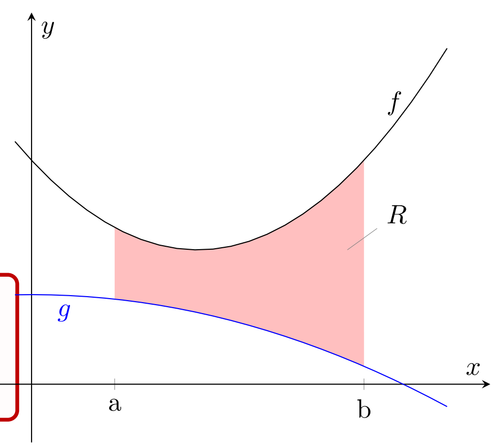
1. If $y = f(x)$ is continuous on $[a, b]$ and $f(x) \geq 0\ \forall x \in [a, b]$, the area of the region under the graph of $f(x)$ from $x = a$ to $x = b$ is given by the integral:

$$A = \int_a^b f(x)\ dx$$



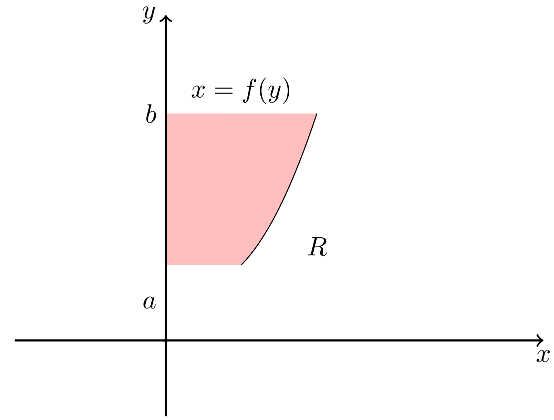
If $f(x)$ and $g(x)$ are continuous and $f(x) \geq g(x)$ for every $x \in [a, b]$, then the area A of the region bounded by the graphs of f and g is given by the integral:

$$A = \int_a^b \big(f(x) - g(x)\big)\ dx$$



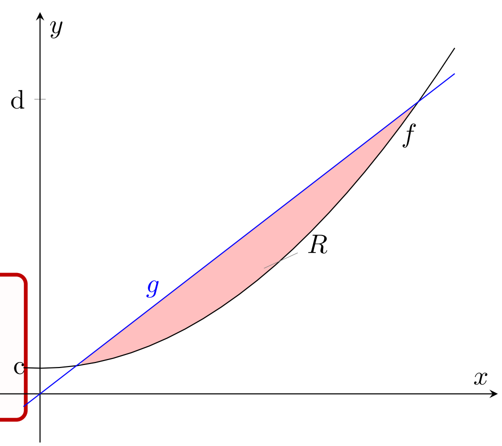
3. If $x = f(y)$ is continuous on $[c, d]$ and $f(y) \geq 0\ \forall y \in [c, d]$, the area of the region under the graph of $f(y)$ from $y = c$ to $y = d$ is given by the integral:

$$A = \int_c^d f(y)\ dy$$



4. If $f(y)$ and $g(y)$ are continuous and $f(y) \geq g(y)$ for every $x \in [c, d]$, then the area A of the region bounded by the graphs of f and g is given by the integral:

$$A = \int_c^d \big(f(y) - g(y)\big)\ dy$$



Example 69 *Sketch the region by the graph of $y = x$ on the interval $[0, 3]$, then find its area.*

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Example 70 *Sketch the region by the graph of* $y = x^2$*, then find its area.*

Example 71 *Sketch the region by the graphs of* $y = x^3$ *and* $y = x$*, then find its area.*

Example 72 *Sketch the region by the graphs of* $x = \sqrt{y}$ *from* $y = 0$ *and* $y = 1$*, then find its area.*

Example 73 *Sketch the region by the graphs of* $y = -x + 6$*,* $y = \sqrt{x}$ *and* $y = 0$*, then find its area.*

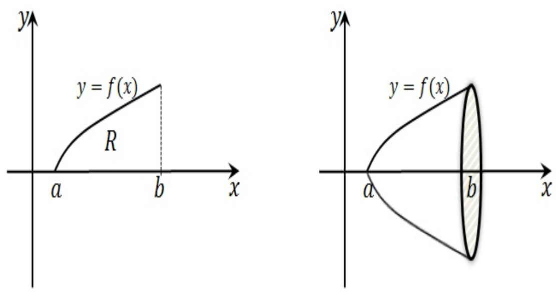
(2) Volumes of Solid of Revolution

In this section, we will start looking at the volume of a solid of revolution. We should first define just what a solid of revolution is.

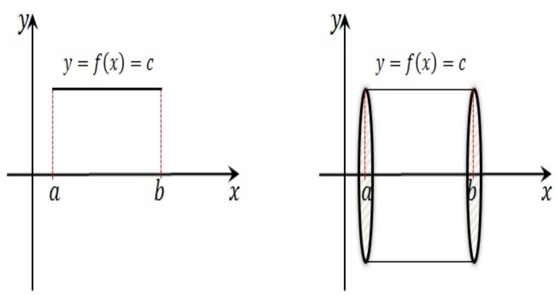
Solid of Revolution

Definition 8 *The solid of revolution (S) is a solid generated from rotating a region R about a line in the same plane where the line is called the axis of revolution.*

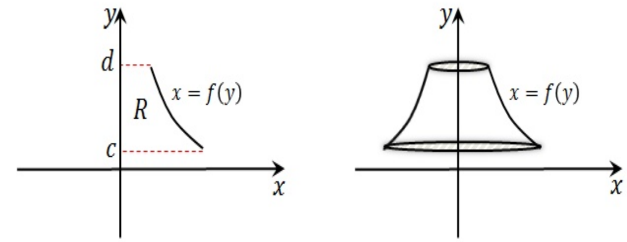
Example 74 *Let $f(x) \geq 0$ be continuous for every $x \in [a,b]$. Let R be a region bounded by the graph of f and x-axis form $x = a$ to $x = b$. Rotating the region R about x-axis generates a solid given in Figure 74.*



Example 75 *Let $f(x)$ be a constant function, as in Figure 75. The region R is a rectangle and rotating it about x-axis generates a circular cylinder.*



Example 76 *Consider the region R bounded by the graph of $f(y)$ from $y = c$ to $y = d$ as in Figure . Revolution of R about y-axis generates a solid of revolution.*



Volumes of Solid of Revolution

One of the simplest applications of integration is to determine a volume of solid of revolution. In this section, we will study three methods to evaluate volumes of revolution known as the disk method, the washer method and the method of cylindrical shells.

(A) Disk Method

Let f be continuous on $[a,b]$ and let R be a region bounded by the graphs, x-axis and the points $x = a$, $x = b$. Let S be a solid generated by revolving R about x-axis. Assume P is a partition of $[a,b]$ and $w_k \in [x_{k-1}, x_k]$ is a marker. For each $[x_{k-1}, x_k]$, we form a rectangle, its high is $f(w_k)$ and its width is Δx_k.

The revolution of the rectangle about x-axis generates a circular disk . Its radius and high are

$$r = f(w_k)\ , \quad h = \Delta x_k\ .$$

The volume of each circular disk is

$$V_k = \pi (f(w_k))^2 \Delta x_k\ .$$

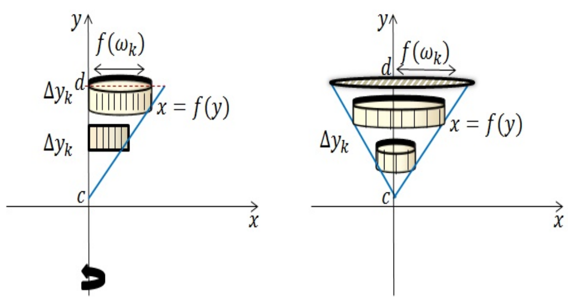
The sum of volumes of the circular disks approximately gives the volume of the solid of revolution:

$$V = \sum_{k=1}^{n} \Delta V_k = \lim_{n\to\infty} \sum_{k=1}^{n} \pi\ (f(w_k))^2\ \Delta x_k = \pi \int_a^b \left[f(x)\right]^2\, dx\ .$$

$$\boxed{V = \pi \int_a^b \left[f(x)\right]^2\, dx\ .}$$

Similarly, we find the volume of the solid of revolution about y-axis. Let f be continuous on $[c,d]$ and let R be a region bounded by the graphs, y-axis and the points $y=c$, $y=d$. Let S be a solid generated by revolving R about y-axis. Assume P is a partition of $[c,d]$ and $w_k \in [y_{k-1}, y_k]$. For each $[y_{k-1}, y_k]$, we form a rectangle, its high is $f(w_k)$ and its width is Δy_k.
Revolution of each rectangle about y-axis generates a circular disk as shown in . Its radius and high are

$$r = f(w_k)\ , \quad h = \Delta y_k\ .$$



The volume of the solid of revolution given in is the sum of volumes of circular disks gives:

$$V = \sum_{k=1}^{n} \Delta V_k = \lim_{n\to\infty} \sum_{k=1}^{n} \pi (f(w_k))^2 \Delta y_k = \pi \int_c^d \left[f(y)\right]^2 \, dy\ .$$

$$\boxed{V = \pi \int_c^d \left[f(y)\right]^2 \, dy\ .}$$

Example 77 *Sketch the region R bounded by the graphs of the equations $y=\sqrt{x}$, $x=4$, $y=0$. Then, find the volume of the solid generated if R is revolved about x-axis.*

Example 78 *Sketch the region R bounded by the graphs of the equations $y=e^x$, $y=e$ and $x=0$. Then, find the volume of the solid generated if R is revolved about y-axis.*

Example 79 *Let $x=y^2$ on the interval $[0,1]$. Rotate the region around the y-axis and find the volume of the resulting solid.*

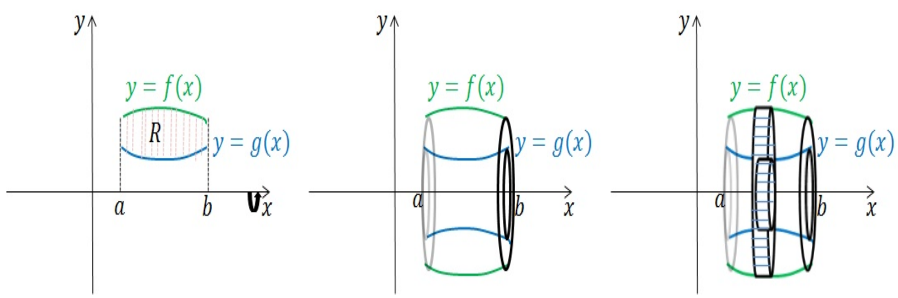
(B) Washer Method

The washer method is the generalization of the disk method for a region between two functions $f(x)$ and $g(x)$. Let R be a region bounded by the graphs of $f(x)$ and $g(x)$ such that $f(x) > g(x)$ from $x = a$ to, $x = b$.

The volume of the solid S generated from rotating the area bounded by the graphs of $f(x)$ and $g(x)$ around x-axis is

$$V = \int_a^b [f(x)]^2 \, dx - \int_a^b [g(x)]^2 \, dx,$$

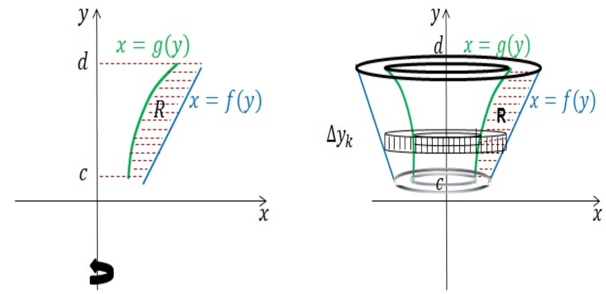
$$\boxed{V = \int_a^b \left([f(x)]^2 - [g(x)]^2\right) \, dx \, .}$$



Similarly, let R be a region bounded by the graphs of $f(y)$ and $g(y)$ such that $f(y) > g(y)$ from $y = c$ to, $y = d$. The volume of the solid S generated from rotating the area bounded by the graphs of f and g around y-axis is

$$V = \int_c^d [f(y)]^2 \, dy - \int_c^d [g(y)]^2 \, dy,$$

$$\boxed{V = \int_c^d \left([f(y)]^2 - [g(y)]^2\right) \, dy \, .}$$



Example 80 *Evaluate the volume of the solid generated by revolution of the bounded region by graphs of the following two functions $y = x^2$ and $y = 2x$ about x-axis.*

Example 81 *Consider a region R bounded by the graphs* $y = \sqrt{x}$, $y = 6 - x$ *and x-axis. Rotate this region about y-axis and find the volume of the generated solid.*

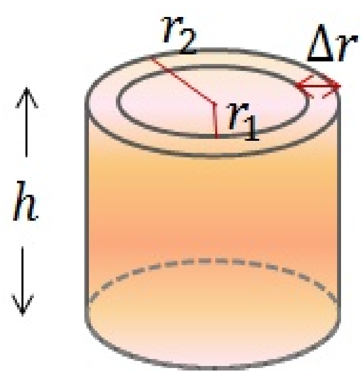
Example 82 *Reconsider the same region as in Example 81 enclosed by the curves* $y = \sqrt{x}$, $y = 6 - x$ *and x-axis. Now rotate this region about the x-axis instead and find the resulting volume.*

(C) Method of Cylindrical Shells

The method of cylindrical shells sometimes easier than the washer method. This is because solving equations for one variable in terms of another is not sometimes simple (i. e., solving x in terms of y and versa visa). For example, the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis. By the washer method, we would have to solve the cubic equation for x in terms of y and this is not simple.

In the washer method, we assume that the rectangle from each sub-interval is vertical to the axis of the revolution, but in the method of cylindrical shells, the rectangle is parallel to the axis of revolution.

As shown in the next figure, let
r_1 be the inner radius of the shell,
r_2 be the outer radius of the shell,
h be high of the shell,
$\Delta r = r_2 - r_1$ be the thickness of the shell,
$r = \frac{r_1+r_2}{2}$ be the average radius of the shell.



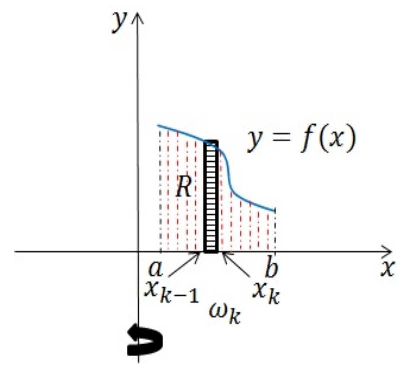
The volume of the cylindrical shell is

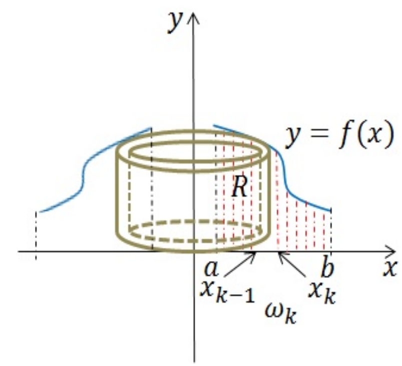
$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi(\frac{r_2 + r_1}{2})h(r_2 - r_1) \\ &= 2\pi r h \Delta r \ . \end{aligned}$$

Now, consider the graph given in the figure below. The revolution of the region R about y-axis generates a solid given in the same figure. Let P be a partition of the interval $[a, b]$ and let w_k be the midpoint of $[x_{k-1}, x_k]$.

The revolution of the rectangle about y-axis generates a cylindrical shell where

the high $= f(w_k)$,
the average radius $= w_k$ and
the thickness $= \Delta x_k$.





Hence, the volume of the cylindrical shell

$$V_k = 2\pi w_k f(w_k)\Delta x_k \ .$$

To evaluate the volume of the whole solid, we sum the volume of all cylindrical shells. This means

$$\boxed{V = \sum_{k=1}^{n} V_k = 2\pi \sum_{k=1}^{n} w_k f(w_k)\Delta x_k \ .}$$

From Riemann sum

$$\sum_{k=1}^{n} w_k f(w_k)\Delta x_k = \int_a^b x f(x)\ dx$$

and this implies

$$\boxed{V = 2\pi \int_a^b x f(x)\ dx \ .}$$

Similarly, if the revolution of the region about x-axis, the volume of the solid of revolution is

$$V = 2\pi \int_c^d y f(y)\ dy \ .$$

Example 83 *Sketch the region R bounded by the graphs of the equations $y = 2x - x^2$ and $x = 0$. Then, by the method of cylindrical shells, find the volume of the solid generated if R is revolved about y-axis.*

Example 84 *Sketch the region R bounded by the graphs of the equations $x = \sqrt{y}$ and $x = 2$. Then, find the volume of the solid generated if R is revolved about x-axis.*

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Homework (5):

Question 1: Sketch the region by the graphs of $y = \sqrt{x}$ and $y = x^2$, then find its area.

Question 2: Sketch the region by the graphs of $y = \sin x$ and $y = \cos x$ on the interval $[0, \frac{\pi}{4}]$, then find its area.

Question 3: Sketch the region by the graphs of $x = y + 2$, $x = y^2$ in the first quadrant, then find its area.

Question 4: Evaluate the volume of the solid generated by revolution of the bounded region by graphs of the following functions $y = -x + 2$, $x = 0$ and $y = 0$ about y-axis.

Question 5: Evaluate the volume of the solid generated by revolution of the bounded region by graphs of the following two functions $y = x^2$ and $y = x$ about x-axis.

Question 6: Evaluate the volume of the solid generated by revolution of the bounded region by graphs of the following two functions $y = x^3$ and $y = x$ from $x = 0$ to $x = \frac{1}{2}$ about x-axis.

(1) Functions of two variables:

Anti-derivatives

Definition 9 *A function of two variables is a rule that assigns an ordered pair (x, y) (in the domain of the function) to a real number w.*

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x, y) \longrightarrow w$$

Example 85 *1.* $f(x, y) = x^2 + y^2$
2. $f(x, y) = \frac{x}{x^2+y^2}$

Partial derivatives of a function of two variables

If $w = f(x, y)$ is a function of two variables, then:
1. The partial derivative of f with respect to x is denoted by $\frac{\partial f}{\partial x}$, $\frac{\partial w}{\partial x}$, f_x or w_x, and it is calculated by applying the rules of differentiation to x and regarding y as a constant.

2. The partial derivative of f with respect to y is denoted by $\frac{\partial f}{\partial y}$, $\frac{\partial w}{\partial y}$, f_y or w_y, and it is calculated by applying the rules of differentiation to y and regarding x as a constant.

Example 86 *Calculate f_x and f_y of the function* $f(x, y) = x^2y^3 + xyln(x + y)$

(2) Functions of three variables :

Anti-derivatives

Definition 10 *A function of two variables is a rule that assigns an ordered pair (x, y) (in the domain of the function) to a real number w.*

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x, y, z) \longrightarrow w$$

Example 87 $f(x, y) = x^2 + y^2 + z$

Partial derivatives of a function of three variables

If $w = f(x, y, z)$ is a function of three variables, then:
1. The partial derivative of f with respect to x is denoted by $\frac{\partial f}{\partial x}$, $\frac{\partial w}{\partial x}$, f_x or w_x, and it is calculated by applying the rules of differentiation to x and regarding y and z as a constant.

2. The partial derivative of f with respect to y is denoted by $\frac{\partial f}{\partial y}$, $\frac{\partial w}{\partial y}$, f_y or w_y, and it is calculated by applying the rules of differentiation to y and regarding x and z as a constant.

3. The partial derivative of f with respect to z is denoted by $\frac{\partial f}{\partial z}$, $\frac{\partial w}{\partial z}$, f_z or w_z, and it is calculated by applying the rules of differentiation to z and regarding x and y as a constant.

Example 88 *Calculate f_x, f_y and f_z of the function* $f(x, y) = z^2y^3 - y^2(x^3 + z)$

Second partial derivatives:
If $w = f(x,y)$ is a function of two variables, then:

1. $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial x} f_x = f_{xx}$
2. $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y} f_y = f_{yy}$
3. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x} f_y = f_{xy}$
4. $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial y} f_x = f_{yx}$

Note: Second partial derivatives of a function of three variables are defined in a same manner.

Anti-derivatives

Theorem 5 *Let $f(x,y)$ be a function of two variables. If f, f_x, f_y, f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.*

Note: If $f(x,y,z)$ is a function of three variables and f has continuous second partial derivatives, then $f_{xy} = f_{yx}, f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$.

Example 89 *Let $f(x,y) = x^3y + xy^2 sin(x+y)$, calculate f_{xy} and f_{yx}.*

Example 90 *Let $f(x,y) = z^3x + y^2(x+yz)$, calculate f_{xy} and f_{yx}.*

1. *f_x, f_y and f_z at $(1,1,1)$.*
2. *f_{xx}, f_{yy} and f_{zz}.*
3. *f_{xy}, f_{yz} and f_{zx} at $(0,-1,1)$.*

===

Chain Rules

1. If $w = f(x, y)$ and $x = g(t)$, $y = h(t)$, such that f, g and h are differentiable then

$$\frac{df}{dt} = \frac{dw}{dt} = \frac{\partial w}{\partial x}\,\frac{dx}{dt} + \frac{\partial w}{\partial y}\,\frac{dy}{dt}$$

2. If $w = f(x, y)$ and $x = g(t, s)$, $y = h(t, s)$, such that f, g and h are differentiable then

$$\frac{\partial f}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial s} = \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial s}$$

3. If $w = f(x, y, z)$ and $x = g(t, s)$, $y = h(t, s)$, $z = k(t, s)$, such that f, g, h and k are differentiable then

$$\frac{\partial f}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial t}$$

$$\frac{\partial f}{\partial s} = \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial s}$$

Example 91 *Let* $f(x, y) = xy + y^2$, $x = s^2t$, *and* $y = s + t$, *calculate*

1. $\frac{\partial f}{\partial t}$ *2.* $\frac{\partial f}{\partial s}$

Example 92 *Let* $f(x, y, z) = x + sin(xy) + cos(xz)$, $x = ts, y = s + t$ *and* $z = \frac{s}{t}$, *calculate*

1. $\frac{\partial f}{\partial t}$ *2.* $\frac{\partial f}{\partial s}$

===

Implicit differentiation

1. Suppose that the equation $F(x,y) = 0$ defines y implicitly as a function of x say $y = f(x)$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

2. Suppose that the equation $F(x,y,z) = 0$ implicitly defines a function $z = f(x,y)$, where f is differentiable, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial x} = -\frac{F_y}{F_z}$$

Example 93 *Let* $y^2 - xy + 3x^2 = 0$, *find* $\frac{dy}{dx}$.

Example 94 *Let* $F(x,y,z) = x^2y + z^2 + sin(xyz) = 0$, *find* $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$.

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Homework (6):

Question 1: If $f(x, y) = 2x^4y^3 - xy^2 + 3y + 1$,

find 1) f_x 2) f_y 3) f_{xy} 4) f_{xx}

Question 2: If $f(x, y) = 4e^{x^2y^3}$,

find 1) f_x 2) f_y 3) f_{xy} 4) f_{xx} 5) f_{yy} at $(1, 1)$.

Question 3: If $x = 2s + t$, $y = s \ln t$, $w = x^2 + 2xy$ at $(1, 1)$
find 1) $\frac{\partial w}{\partial s}$ 2) $\frac{\partial w}{\partial s}$ at $(1, 1)$.

Question 4: By using the implicit function differentiation, find $\frac{dy}{dx}$

1) $x^3 - 3xy^2 + y^3 = 5$

2) $x - \sqrt{xy} + 3y = 4$

3) $2x^3 + x^2y + y^3 = 1$

Definition of a differential equation

Definition 11
An equation that involves $x, y, y', y'', y''', y^{(4)}, ..., y^{(n)}$ for a function $y(x)$ with n^{th} derivative $y^{(n)}$ of y with respect to x is an ordinary differential Equation of order n.

Example 95 *1. $y' = x^2 + 5$ is a differential equation of order 1.*
2. $y'' + x(y')^3 - y = x$ is a differential equation of order 2.
3. $(y^{(4)})^3 + x^2y'' = 2x$ is a differential equation of order 4.

Remark 3 *$y = y(x)$ is called a solution of a differential equation if $y = y(x)$ satisfies that differential equation.*

Example 96 *Consider the differential equation $y' = 6x + 4$, then $y = 3x^2 + 4x$ is a solution of that differential equation.*

Note:
1. $y = 3x^2 + 4x$ is the general solution of that differential equation.

2. If an initial condition was added to the differential equation to assign a certain value for c then $y = y(x)$ is called the particular solution of the differential equation.

Example 97 *Consider the differential equation $y' = 6x + 4$ with the initial condition $y(0) = 2$, $y = 3x^2 + 4x + c$ is the general solution of the differential equation,*

$$y(0) = 2 \Rightarrow 3(0)^2 + 4(0) + c = 2 \Rightarrow c = 2 \ .$$

Hence $y = 3x^2 + 4x + 2$ is the particular solution of the differential equation.

Separable Differential equations

The separable differential equation has the form

$$M(x) + N(y)y' = 0$$

where $M(x)$ and $N(y)$ are continuous functions.
To solve the separable differential equation :
1. Write the equation as $M(x)dx + N(y)dy = 0 \Rightarrow N(y)dy = -M(x)dx$.

2. Integrate the left-hand side with respect to y and the right-hand side with respect to x .

Example 98 *Solve the differential equation $y' + y^3e^x = 0$.*

Example 99 *Solve the differential equation* $\frac{dy}{dx} = y^2 e^x$, $y(0) = 1$

Example 100 *Solve the differential equation* $dy - sinx(1 + y^2)dx = 0$
.

Example 101 *Solve the differential equation* $e^{-y} sinx - y' cos^2 x = 0$

Example 102 *Solve the differential equation* $y' = 1 - y + x^2 - yx^2$.

The first-order linear differential equation has the form

$$y' + P(x)y = Q(x) \ ,$$

where $P(x)$ and $Q(x)$ are continuous functions of x

To solve the first-order linear differential equation :
1. Compute the integrating factor $u(x) = e^{\int P(x)\ dx}$.

2. The general solution of the first-order linear differential equation is

$$y(x) = \frac{1}{u(x)} \int u(x)Q(x)\ dx$$

Example 103 *Solve the differential equation* $x\frac{dy}{dx} + y = x^2 + 1$.

Example 104 *Solve the differential equation* $y' - \frac{2}{x}y = x^2 e^x$, $y(1) = e$
.

Example 105 *Solve the differential equation* $y' + y = cos(e^x)$.

Example 106 *Solve the differential equation* $xy' - 3y = x^2$.

Homework (7):

Question 1: Solve the differential equations:

1) $x^2dy + y^2dx = 0$

2) $\cos^2 x dy - y^2dx = 0$

3) $x\frac{dy}{dx} - 2y = x^3 secx \tan x$

4) $y' = x^2 + y^2$

5) $y' + 3y = e^{-2x}$

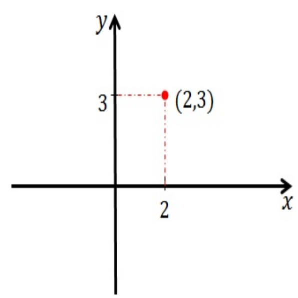
Question 2: Solve the differential equation $y' + 2y = x$, $y(0) = 1$

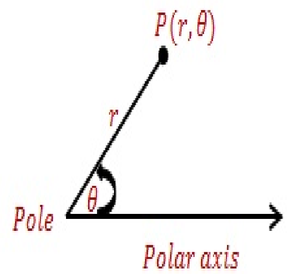
Question 3: Solve the differential equation $xy' + y = \sin x$, $y(\frac{\pi}{3}) = 2$

Question 4: Solve the differential equation $\frac{dy}{dx} + y - \frac{1}{e^x+1} = 0$

(1) Definition:

Definition 12 *The polar coordinate system is a two-dimensional coordinate system in which each point P on a plane is determined by a distance r from a fixed point O that is called the pole (or origin) and an angle θ from a fixed direction.*





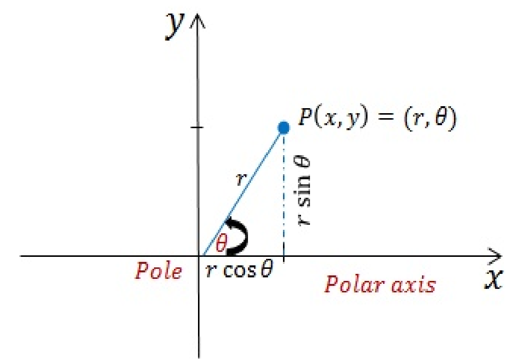
Remark 4
In the polar coordinates (r,θ), if $r>0$, the point (r,θ) lies in the same quadrant as θ; if $r<0$, it lies in the quadrant on the opposite side of the pole. Meaning that, the polar coordinates (r,θ) and $(-r,\theta)$ lie in the same line through the pole O and at the same distance $\mid r \mid$ from O, but on opposite sides of O.

Example 107 *Plot the points whose polar coordinates are given:*

1. $(1,\pi/4)$
2. $(2,3\pi)$
3. $(2,-\pi/3)$
4. $(-3,\pi/6)$

(2) Relationship between Rectangular and Polar Coordinates

Let (x,y) be a rectangular coordinate and (r,θ) be a polar coordinate. Let the pole at the origin point and polar axis on x-axis, and the line $\theta=\frac{\pi}{2}$ on y-axis as shown in Figure .



From the triangle OAP

$$\cos\theta=\frac{x}{r}\Rightarrow x=r\cos\theta\ ,\quad \sin\theta=\frac{y}{r}\Rightarrow y=r\sin\theta\ .$$

Thus,

$$\begin{aligned} x^2+y^2&=(r\cos\theta)^2+(r\sin\theta)^2,\\ &=r^2(\cos^2\theta+\sin^2\theta)\\ x^2+y^2&=r^2\ . \end{aligned}$$

$$\boxed{\begin{aligned} x\ =\ r\cos\theta,\quad y\ =\ r\sin\theta\\ \tan\theta=\frac{y}{x},\quad x^2+y^2\ =\ r^2 \end{aligned}}$$

Example 108 *Convert the points from the polar coordinates to the rectangular coordinates:*

1. $(1,\pi/4)$
2. $(2,\pi)$

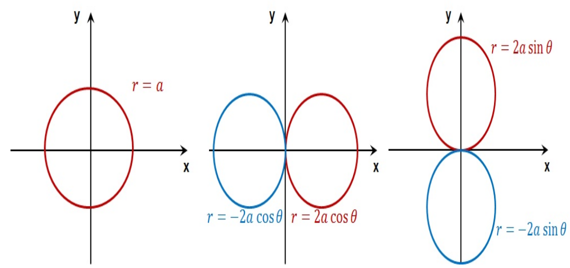
Example 109 *Convert the points from the rectangular coordinates to polar coordinates:*

1. $(5, 0)$ *2.* $(2\sqrt{3}, 2)$

(3) Sketch of Polar Curves

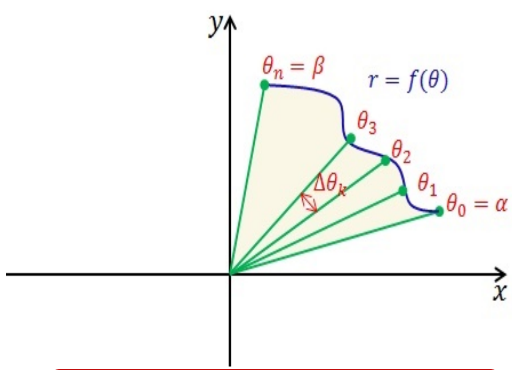
Circles in polar coordinates

1. A circle its center at O and radius a: $r = a$.
2. A circle its center at $(a, 0)$ and radius $|a|$: $r = 2a\cos\theta$.
3. A circle its center at $(0, a)$ and radius $|a|$: $r = 2a\sin\theta$.



(4) Area in Polar Coordinates

Let $r = f(\theta)$ be a continuous function on the interval $[\alpha, \beta]$ such that $0 \le \alpha \le \beta \le 2\pi$. Let $f(\theta) > 0$ over that interval and R be a polar region bounded by the polar equations $r = f(\theta)$, $\theta = \alpha$ and $\theta = \beta$ as shown in Figure .



$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} \left(f(\theta)\right)^2 \, d\theta}$$

Similarly, assume f and g are continuous on the interval $[\alpha, \beta]$ such that $f(\theta) > g(\theta)$. The area bounded by the curves of f and g on the interval $[\alpha, \beta]$ is

$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} \left[\left(f(\theta)\right)^2 - \left(g(\theta)\right)^2\right] d\theta}$$

Example 110 *Find the area of the region bounded by the graph of the polar equation $r = 2$.*

Example 111 *Find the area of region lying in the first quadrant and inside the circle with polar equation $r = 2$.*

Example 112 *Find the area of region lying inside the circle with polar equation $r = 2$ and outside the circle with polar equation $r = 1$.*

===

Homework (8):

Question 1: Convert the points from the polar coordinates to the rectangular coordinates:

1. $(3, \pi/2)$
2. $(2, \frac{\pi}{3})$

Question 2: Convert the points from the rectangular coordinates to the polar coordinates:

1. $(4, 0)$
2. $(6, 3)$

Question 3: Find the area of region lying inside the circle with polar equation $r = 2$.

Question 4: Find the area of region lying in the **first quadrant** and inside the circle with polar equation $r = 3$.

Question 5: Find the area of region lying inside the circle with polar equation $r = 1$ and outside the circle with polar equation $r = 2$.

Question 6: Find the area of region lying in the **first quadrant** and inside the circle with polar equation $r = 2$ and outside the circle with polar equation $r = 3$.

==

CHAPTER 1:

1. $F(1,1)$, $D: y=-3$
2. $(y-2)^2 = 2(x+4)$
3. $(x-3)^2 = 8(y-4)$
4. $\frac{x^2}{16}+\frac{y^2}{9}=1$
5. $\frac{(x-7)^2}{25}+\frac{(y+2)^2}{16}=1$
6. $\frac{y^2}{4}-\frac{x^2}{9}=1$
7. $\frac{(x-7)^2}{1}-\frac{(y+2)^2}{8}=1$
8. $\frac{(x+3)^2}{5}+\frac{(y-4)^2}{1}=1$
9. $(x+1)^2=-2(y-\frac{3}{2})$
10. $\frac{(y+3)^2}{5}-\frac{(x+4)^2}{1}=1$

CHAPTER 2:

Q3: 1) 11 2) 73

CHAPTER 3:

Question 1:

1. X = $\begin{pmatrix}5\\6\\7\end{pmatrix}$ 2. X = $\begin{pmatrix}6\\8\\10\end{pmatrix}$

Question 2: 1. X = $\begin{pmatrix}5\\6\\7\end{pmatrix}$ 2. X = $\begin{pmatrix}6\\4\\2\end{pmatrix}$

Question 3: 1. X = $\begin{pmatrix}7/2\\-7/2\\7\end{pmatrix}$ 2. X = $\begin{pmatrix}6\\8\\10\end{pmatrix}$

CHAPTER 4:

Question 1:

1. $\frac{\tan(3x-5)}{3}+c$
2. $\sin^{-1}(\frac{x}{4})+c$
3. xe^x-e^x+c
4. $x\sin x+\cos x+c$
5. $xsin^{-1}x+\sqrt{1-x^2}+c$
6. $\frac{1}{3}\ln|x-2|-\frac{1}{3}\ln|x+1|+c$
7. $\frac{(2x^2-3)^9}{36}+c$
8. $3\sin\sqrt[3]{x}+c$

Question 2:

1. $\tan x+secx+c$
2. $\frac{x^2(\ln x-1)}{8}+c$
3. $\frac{32}{3}$
4. $\ln 3-\ln(e-2)$
5. $2\ln 7-\ln 4$
6. $-3\frac{1}{2}$

CHAPTER 5:

Q1. $\frac{5}{12}$ Q2. $\frac{2-\sqrt{2}}{\sqrt{2}}$

Q3. $\frac{27}{6}$

Q4. $\frac{64\pi}{3}$ Q5. $\frac{2\pi}{15}$

CHAPTER 6:

Q2: 1) $8e$ 2) $12e$ 3) $48e$ 4) $24e$ 5) $60e$

Q3: 1) 12 2) 12

Q4: 1) $-\frac{3x^2-3y^2}{-6xy+3y^2}$ 2) $-\frac{2\sqrt{xy}-y}{-x+6\sqrt{xy}}$ 3) $-\frac{6x^2+2xy}{x^2+3y^2}$

CHAPTER 7:

Q1: 1) $y=\frac{-x}{1+cx}$ 2) $y=\frac{-1}{c+\tan x}$

Q4: $y=\frac{1}{e^x}\ln(e^x+1)+c$

CHAPTER 8:

Q1: 1) $(0,3)$

Q2: 2) $(2,\frac{\pi}{6})$

Q6: $\frac{5\pi}{2}$