PHYS-505/551
Scattering Theory-b

Lecture-10

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Scattering in low energies: The method of partial waves-a

- So far we considered approximating solution to the problem of calculation of the differential scattering cross-section. Where the interaction potential is considered small compared to the kinetic energy of the incident particle.
- The method of partial waves is an exact method which is practically used as an approximation method in low energies.

Scattering in low energies: The method of partial waves-b

- The mathematical form of the wave function at long distances :

$$\psi \to e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \qquad (10.1)$$

can be expanded in terms of Legendre polynomials $P_l(\cos\theta)$

$$\psi(r,\theta) = \sum_{l=0}^{\infty} A_l \frac{\chi_l(r) P_l(\cos\theta)}{r} \qquad (10.2)$$

where the coefficient A_l and the functions χ_l are to be determined

Scattering in low energies: The method of partial waves-c

- The functions χ_l satisfy the radial Schrödinger eq.

$$\left[\frac{d}{dr^2} + k^2 - V(r) - \frac{l(l+1)}{r^2}\right]\chi_l(r) = 0 \qquad (10.3)$$

and the following boundary conditions:

$$\chi_l(0) = 0$$

Distorted wave function

$$\chi_l(\infty) \to \left[A_l \underbrace{j_l(kr)}_{\substack{spherical \\ Bessel\ function}} + B_l \underbrace{n_l(kr)}_{\substack{spherical \\ Neumann\ function}}\right] r = \frac{1}{k} C_l \sin\left(kr - \frac{\pi l}{2} + \delta_l\right)$$

(10.4)

Scattering in low energies: The method of partial waves-d

- What about the parameter δ_l ?
- It is called the *phase shift* since it determines the difference in phase between this solution and the solution of the free radial equation:

$$\chi_l(\infty) = \frac{1}{k} C_l \sin\left(kr - \frac{\pi l}{2}\right) \qquad (10.5)$$

- δ_l is a real value which vanishes for all values of l in the absence of a scattering potential

Scattering in low energies: The method of partial waves-e

- The following relations can be proved

$$A_l = (2l+1) i^l e^{i\delta_l} \qquad (10.6a)$$

$$f_k(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l P_l(\cos\theta) \qquad (10.6b)$$

$$d\sigma / d\Omega = \frac{1}{k^2} \left| \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l P_l(\cos\theta) \right|^2 \qquad (10.6c)$$

$$\sigma_T = 2\pi \int_0^{\pi} |f(\theta)|^2 \sin\theta \, d\theta = \sum_{l=0}^{\infty} \sigma_l = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l \qquad (10.6d)$$

Comments

- The quantities σ_l are called **partial cross sections**. They correspond to the scattering of particles in various angular momentum states.
- The differential cross section consists of a superposition of terms with different angular momenta. This gives rise to **interference patterns** between different partial waves corresponding to different values of l.
- The interference terms go away in the total cross section when the integral over θ is carried out.

The Optical Theorem

- From the above relations we can derive that:

$$\sigma_T = \frac{4\pi}{k}\operatorname{Im}\left(f_k\left(0\right)\right) \qquad (10.7)$$

- This is the so-called *Optical Theorem.* This theorem has a very deep physical meaning: When a beam of particles is scattered, it is obvious that. since the scattered particles are removed from the beam, the intensity of the beam, after the scattering region, will be smaller. But quantum scattering is purely a wave effect. This decrease at the incidence direction (θ=0), is the result of the destructive interference of the two terms of (6.1) between the incident and the scattered wave. The Optical theorem is the result of the conservation of particles or, equivalently, the conservation of probability. Optical theorem holds much more generally: when f depends on ϕ as well as on θ, and when we include inelastic scattering and absorption as well as inelastic scattering.

Scattering in low energies: The method of partial waves-f

- For the phase shifts in the case where the potential vanishes outside a region $r < a$, and since the radial wave-function $R_l = \chi_l(r)/r$ and its derivative are continuous at the boundary $r = a$, we have:

$$\tan\delta_l = \frac{kj_l'(ka) - \gamma_l j_l(ka)}{kn_l'(ka) - \gamma_l n_l(ka)} \qquad (10.8)$$

Where $\gamma_l = \frac{1}{R_l}\frac{dR_l}{dr}\Big|_{r=a}$ $\quad [R_l(r) = \chi_l(r)/r]$

The method of phase shifts as a low energy approximation-a

- The method of phase shifts is an **exact** method of solving the problem of scattering in a central potential.
- But life is more complicated
- First, the radial equation rarely can be solved exactly, specially in cases where $l \neq 0$.
- Second, even in the case of a solution, we would have to sum an infinite series of partial waved to calculate the scattering amplitude $f(\theta)$.

The method of phase shifts as a low energy approximation-b

- All the above mean that this method is limited in the cases where only the first partial waves ($l = 0$ or 1) have a significant contribution in the scattering amplitude.
- This is what happens in the case of a *low energy scattering* inside a *short range potential* like the strong interaction potential between nucleons.

The method of phase shifts as a low energy approximation-c

- How many partial waves contribute to the scattering amplitude?
- This can be estimated by the inequality:

$$l \leq ka = 2\pi\left(a / \lambda\right) \qquad (10.9)$$

where a is the range of the potential and λ the wavelength of the incident particles.

The method of phase shifts as a low energy approximation-d

- For wavelengths far larger than the range of the potential ($\lambda >> a$) the only partial wave that participates in the scattering is the $l = 0$ or the *s-wave*. In the scattering amplitude does not depend on θ and we have a *purely isotropic scattering*

$$f_k(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left(e^{2i\delta_l}-1\right)P_l(\cos\theta) =$$

$$= \frac{3}{2ik}\left(e^{2i\delta_0}-1\right)P_0(\cos\theta) \underbrace{=}_{P_0(\cos\theta)=1} \frac{3}{2ik}\left(e^{2i\delta_0}-1\right) \qquad (10.10)$$

The method of phase shifts as a low energy approximation-e

- The existence of the first deviations from the isotropic distribution is a sign of the presence of the *p-wave* ($l = 1$) in the scattering amplitude.

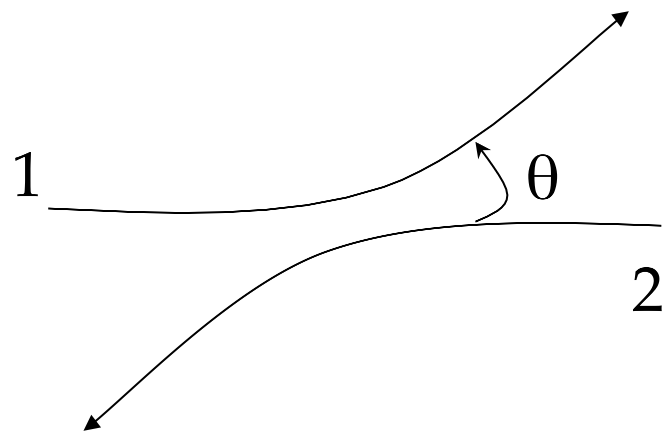
$$f_k(\theta) = f_0 + f_1 \cos\theta$$

- The task for an experimentalist is to find the proper potential which fits better to the experimental curves $f_0(E)$ and $f_1(E)$

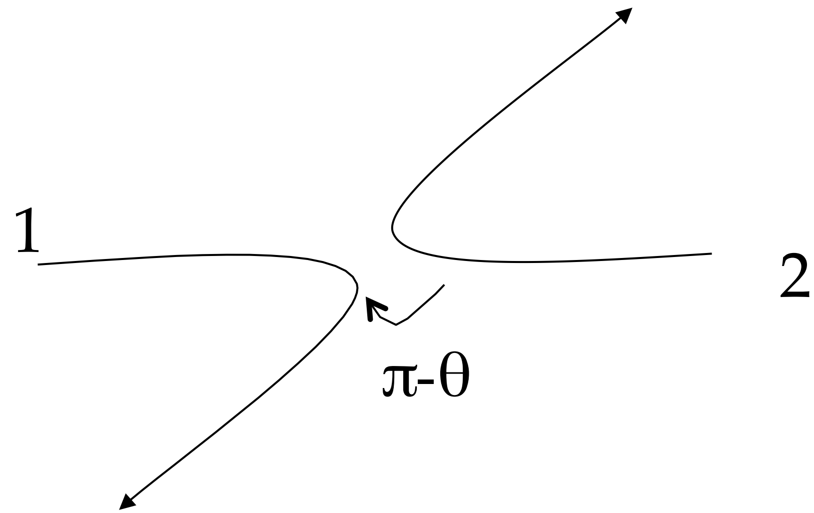
Scattering of identical particles-a

- The scattering between *two identical particles* is a very interesting quantum mechanical problem. It becomes more clear if we decide to study the effect in the system of the center of mass (CM) of the two particles where the distinction between a particle-target and a particle-projectile is not valid.
- We give a schematic representation in the next transparency

Scattering of identical particles-b



$amplitude : f(\theta)$



$amplitude : f(\pi - \theta)$

Scattering of two identical particles at the center of mass system. If we place a particle detector it cannot distinguish if the particle which arrives is the particle 1 or 2. This means that we must add the scattering amplitudes.

Scattering of identical particles-c

- The scattering amplitude is given by the expression:

$$f(\theta) \pm f(\pi - \theta) \qquad (10.11)$$

- Where the proper choice of sign depends on the spin of particles and their relative orientation.

Scattering of identical particles-d

- The simplest case is that of two particles with spin equal to zero (so they are bosons). In this case, since, the wave-function $\psi\left(\mathbf{r}_1, \mathbf{r}_2\right)$ must be symmetric to the interchange 1 <--> 2, and because this change is equivalent to θ <--> π-θ then the correct choice for the sign + and the correct expression for the differential cross-section will be:

$$\frac{d\sigma}{d\Omega} = \left| f\left(\theta\right) + f\left(\pi - \theta\right) \right|^2 \qquad (10.12)$$

Scattering of identical particles-e

- The previous expression can be analytically expressed as:

$$\frac{d\sigma}{d\Omega} = \left|f(\theta)\right|^2 + \left|f(\pi-\theta)\right|^2 + f^*(\theta)f(\pi-\theta) + f(\theta)f^*(\pi-\theta)$$

- The first two terms represent the differential cross section in the case where the two particles were *distinguishable*, the other two terms represent the *interference terms* because the two particles are *indistinguishable.*

Scattering of identical particles-f

- The difference between the two cases is more clear when the scattering is at $\theta = 90^0$. We can show that:

$$\left(\frac{d\sigma}{d\Omega}\right)_{indistinguishable} = 4\left|f\left(\pi/2\right)\right|^2$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{distinguishable} = 2\left|f\left(\pi/2\right)\right|^2$$

Scattering of identical particles-g

- In the case where the two particles have a spin equal to 1/2 (like electrons or protons) then the relative spin orientation plays an important role.
- If their spins are parallel (*S*=1, triplet state) then the spatial wave-function will be anti-symmetric with respect to the interchange 1 <--> 2. In this case we must chose the - sign and

$$\left(\frac{d\sigma}{d\Omega}\right)_{\uparrow\uparrow} = \left|f(\theta) - f(\pi - \theta)\right|^2$$

- If their spins are anti-parallel (*S*=0,singlet state) then the spatial wave-function will be symmetric with respect to the interchange 1 <--> 2. In this case we must chose the + sign and

$$\left(\frac{d\sigma}{d\Omega}\right)_{\uparrow\downarrow} = \left|f(\theta) + f(\pi - \theta)\right|^2$$

Scattering of identical particles-h

- The result is really impressing!
- Although spin does not affect the electrostatic force between the two particles, the relative orientation of the two spins affects dramatically their scattering (for example you can check what happens when the spins are parallel and the scattering is at 90 degrees).

Scattering of identical particles-i

- For an unpolarized beam of particles with spin s, the system can be in $\left(2s+1\right)^2$ states that are distributed with equal probabilities. From the total number of possibilities, $(2s+1)$ spin states are anti-symmetric. Therefore, the differential cross section is

$$\frac{d\sigma}{d\Omega} = \left|f\left(\theta\right)\right|^2 + \left|f\left(\pi-\theta\right)\right|^2 + \frac{\left(-1\right)^{2s}}{2s+1} 2\,\mathrm{Re}\left[f\left(\theta\right)f^*\left(\pi-\theta\right)\right] \qquad (10.13)$$