Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS
MATH 1110

Instructor:
Dr. Mohamed El-Shazly
Assistant Prof. of Mechanical Design and Tribology
mohamed.elshazly@ams-sae.com
Office: S053

Example 1

SINE & COSINE INTEGRALS

Evaluate $\int \cos^3 x \, dx$

– Simply substituting $u = \cos x$ isn't helpful, since then $du = -\sin x \, dx$.

– In order to integrate powers of cosine, we would need an extra $\sin x$ factor.

– Similarly, a power of sine would require an extra $\cos x$ factor.

Example 1

SINE & COSINE INTEGRALS

Thus, here we can separate one cosine factor and convert the remaining $\cos^2 x$ factor to an expression involving sine using the identity $\sin^2 x + \cos^2 x = 1$:

$$\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x) \cos x$$

Example 1

SINE & COSINE INTEGRALS

We can then evaluate the integral by substituting $u = \sin x$.

So, $du = \cos x\, dx$ and

$$\begin{aligned}
\int \cos^3 x\, dx &= \int \cos^2 x \cdot \cos x\, dx \\
&= \int (1 - \sin^2 x) \cos x\, dx \\
&= \int (1 - u^2)\, du = u - \tfrac{1}{3}u^3 + C \\
&= \sin x - \tfrac{1}{3}\sin^3 x + C
\end{aligned}$$

Combinations of sin, cos

- General form $$\int \sin^m x \cdot \cos^n x \, dx$$
- If either *n* or *m* is odd, use techniques as before
  - Split the odd power into an even power and power of one
  - Use Pythagorean identity
  - Specify *u* and *du,* substitute
  - Usually reduces to a polynomial
  - Integrate, un-substitute

Example 2

SINE & COSINE INTEGRALS

Find $\int \sin^5 x \cos^2 x \, dx$

– We could convert $\cos^2 x$ to $1 - \sin^2 x$.

– However, we would be left with an expression in terms of $\sin x$ with no extra $\cos x$ factor.

Example 2

SINE & COSINE INTEGRALS

$$\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \sin x$$
$$= (1 - \cos^2 x)^2 \cos^2 x \sin x$$

Example 2

Substituting *u* = cos *x*, we have *du* = sin *x* *dx*.

So,

$$\int \sin^5 x \cos^2 x\, dx = \int (\sin^2 x)^2 \cos^2 x \sin x\, dx$$

$$= \int (1-\cos^2 x)^2 \cos^2 x \sin x\, dx = \int (1-u^2)^2 u^2 (-du)$$

$$= -\int (u^2 - 2u^4 + u^6) du = -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} \right) + C$$

$$= -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C$$

EXAMPLE 4:

Find $\int cos^3 x\, sin^3 x\, dx.$

$$\int cos^2 x\, sin^3 x\, cos\, x\, dx =$$

$$\int (1 - sin^2 x) sin^3 x\, cos\, x dx$$

$$= \int (sin^3 x - sin^5 x) cos\, x\, dx\, .$$

Let $u = sin\, x \;\; \Rightarrow \;\; du = cos\, x\, dx\, .$

$$\int \left(u^3 - u^5\right) du = \frac{u^4}{4} - \frac{u^6}{6} = \frac{1}{4} sin^4 x - \frac{1}{6} sin^6 x + C$$

EXAMPLE 5:

Find $\int cos^{-4}x\, sin^3 x\, dx$.

$$\int cos^{-4}x\, sin^2 x\, (sin\, x) dx =$$

$$\int cos^{-4}x \left(1 - cos^2 x\right) sin\, x\, dx =$$

where $u = cos\, x$, $du = -sin\, x\, dx$.

$$\int cos^{-4}x \left(1 - cos^2 x\right) sin\, x\, dx =$$

$$-\int u^{-4}(1 - u^2)(-sin\, x)dx =$$

$$-\int \left(u^{-4} - u^{-2}\right) du$$

$$-\int u^{-4}(1 - u^2)(-sin\, x)dx =$$

$$-\int \left(u^{-4} - u^{-2}\right) du$$

$$= \int \left(u^{-2} - u^{-4}\right) du = \frac{u^{-1}}{-1} - \frac{u^{-3}}{-3} =$$

$$-\frac{1}{cos\, x} + \frac{1}{3\, cos^3 x} + C.$$

EXAMPLE 6:

Find $\displaystyle\int cos^4 x\, dx.$

Use $cos^2 x = \dfrac{1 + cos\, 2x}{2}$.

$$\int cos^4 x\, dx = \int \left(\frac{1 + cos\, 2x}{2}\right)^2 dx =$$

$$\int \frac{1 + 2\, cos\, 2x + cos^2 2x}{4} dx =$$

$$\int \left(\frac{1}{4} + \frac{cos\, 2x}{2} + \frac{cos^2 2x}{4}\right) dx =$$

$$\frac{x}{4} + \frac{sin\, 2x}{4} + \frac{1}{4}\int \frac{1 + cos\, 4x}{2} dx =$$

$$\frac{x}{4} + \frac{sin\, 2x}{4} + \frac{1}{8}x + \frac{1}{32} sin\ 4x =$$

$$\boldsymbol{\frac{3x}{8} + \frac{sin\, 2x}{4} + \frac{sin\, 4x}{32} + C.}$$

EXAMPLE 7:

Find $\int sec^4 x\, tan^{10} x\, dx$

We will let $u = tan\ x$, $du = sec^2 x\, dx$, and use the identity

$$1 + tan^2 x = sec^2 x\ .$$

Peel off a factor of $sec^2 x$.

$$\int sec^4 x\, tan^{10} x\, dx = \int sec^2 x\, tan^{10} x\, sec^2 x\, dx =$$

$$\int (1 + tan^2 x) tan^{10} x\, sec^2 x\, dx =$$

$$\int \left(tan^{10} x + tan^{12} x\right) sec^2 x\, dx =$$

$$\int (u^{10} + u^{12}) du =$$

$$\frac{tan^{11} x}{11} + \frac{tan^{13} x}{13} + C\ .$$

EXAMPLE 8:

Find $\int sec^3 x\, tan^3 x\, dx.$

Technique: Peel off a $\boldsymbol{sec\, x\, tan\, x}$.

Then let

$u = sec\, x,\ du = sec\, x\, tan\, x\, dx,$

and use the identity $1 + tan^2 x = sec^2 x.$

$$\int sec^3 x\, tan^3 x\, dx =$$

$$\int sec^2 x\, tan^2 x\, (\boldsymbol{sec\, x\, tan\, x}) dx$$

$u = sec\, x,\ du = sec\, x\, tan\, x\, dx$

$tan^2 x = sec^2 x - 1 = u^2 - 1$

$$\int sec^2 x\, sec\, x\, tan^2 x\, tan\, x\, dx =$$

$$\int sec^2 x \left(sec^2 x - 1\right) sec\, x\, tan\, x\, dx$$

$$= \int \left(sec^4 x - sec^2 x\right)(sec\, x\, tan\, x\,)dx =$$

$$\int \left(u^4 - u^2\right) du$$

(where $u = sec\, x$, $du = sec\, x\, tan\, x\, dx$)

$$= \frac{u^5}{5} - \frac{u^3}{3} = \frac{sec^5 x}{5} - \frac{sec^3 x}{3} + C\,.$$

EXAMPLE 9:

Find $\int sec^6x\, tan^5x\, dx.$

Now we have a choice.
We can keep one sec^2x , use the identity
$sec^2x = 1 + tan^2x$, and substitute $u = tan\, x$.

$$\int sec^2x\, sec^4x\, tan^5x\, dx =$$

$$\int \left(1 + tan^2x\right)^2 tan^5x\, sec^2x\, dx =$$

$$\int \left(1 + u^2\right)^2 u^5 du = \int \left(1 + 2u^2 + u^4\right) u^5 du =$$

$$\int \left(u^5 + 2u^7 + u^9\right) du = \frac{u^6}{6} + \frac{2u^8}{8} + \frac{u^{10}}{10} =$$

$$\frac{tan^6x}{6} + \frac{tan^8x}{4} + \frac{tan^{10}x}{10} + C\,.$$

EXAMPLE 10:

Find $\int sec^6 x\, tan^5 x\, dx$ a 2nd way.

We can peel off a $sec\, x\, tan\, x$ and make the substitution $u = sec\, x,\ du = sec\, x\, tan\, x\, dx$.

$$\int sec^6 x\, tan^5 x\, dx =$$

$$\int sec^5 x\, tan^4 x\, (sec\, x\, tan\, x\, dx) =$$

$$\int sec^5 x (sec^2 x - 1)^2 (sec\, x\, tan\, x\, dx) =$$

$$\int u^5 (u^2 - 1)^2 du = \int u^5 \left(u^4 - 2u^2 + 1\right) du =$$

$$\int \left(u^9 - 2u^7 + u^5\right) du =$$

$$\frac{sec^{10} x}{10} - \frac{sec^8 x}{4} + \frac{sec^6 x}{6} + C\,.$$

EXAMPLE 11:

Find $\int csc^4x\, cot^6x\, dx$.

Let $u = cot\ x,\ du = -csc^2x\, dx,$
$$1 + cot^2x = csc^2x\,.$$

$$\int cot^6x\, csc^2x\, csc^2x\, dx =$$
$$-\int u^6(1+u^2)du =$$
$$-\int (u^6+u^8)du = -\frac{u^7}{7} - \frac{u^9}{9} =$$
$$-\frac{cot^7x}{7} - \frac{cot^9x}{9} + C\,.$$

EXAMPLE 12:

Find $\int tan^2x\, dx$.

Use $tan^2x = sec^2x - 1$.

$$\int tan^2x\, dx = \int (sec^2x - 1) dx =$$

$$\int sec^2x\, dx - \int 1\, dx =$$

$$tan\, x - x + C\, .$$

EXAMPLE 13:

find $\int tan^4x\, dx.$

$$\int tan^4x\, dx = \int tan^2x\, tan^2x\, dx =$$

(use $tan^2x = sec^2x - 1$)

$$\int (sec^2x - 1)tan^2x\, dx =$$

$$\int sec^2x\, tan^2x\, dx - \int tan^2x\, dx =$$

$$\frac{tan^3x}{3} - (tan\, x - x) =$$

$$\frac{tan^3x}{3} - tan\, x + x + C\, .$$