Phys 505 Lecture 12
Quantum Simple Harmonic Oscillator

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Dirac formalism: a new way for representing wavefunctions-a

- According to Dirac, any quantum state ψ is represented by two vectors: The first is a column vector, is denoted as $\left|\psi\right\rangle$, and is called ***ket vector***. The second is a row vector and is denoted by $\left\langle\psi\right|$ and is called ***bra vector***. These names come from the english word bracket because in this formalism the dot product of two states ψ and φ is given by

$$\left(\psi,\phi\right)=\left\langle\psi\middle|\varphi\right\rangle$$

Dirac formalism: a new way for representing wavefunctions-b

- With this formalism the average value of a physical quantity on a state ψ is denoted by:

$$\langle A \rangle = \int_{-\infty}^{\infty} \psi^*(x)\big(A\psi(x)\big)dx = \langle \psi | A | \psi \rangle$$

- The two vectors are related by the following relations

$$\big(|\psi\rangle\big)^\dagger = \langle \psi |, \quad \big(\langle \psi |\big)^\dagger = |\psi\rangle$$

$$\big(c_1|\psi_1\rangle + c_2|\psi_2\rangle\big)^\dagger = c_1^*\langle \psi_1| + c_2^*\langle \psi_2|$$

…the quantum SHO…

- Since the quantum SHO has equidistant energy eigenvalues which are produced by the ground state energy by "ascending" at a constant step $\hbar\omega$, it is reasonable to think if we could do the same for the SHO eigenfunctions.

…the algebraic solution-a…

- Consider the ket notation for the SHO eigenfunctions:

$$\psi_n(x) \rightarrow \left| n \right\rangle$$

- Consider also the following raising and lowering operators a and $a^\dagger$

These operators act as follows on a certain eigenstate of the SHO

…the algebraic solution-b…

$$a^{\dagger}\left|n\right\rangle \rightarrow \left|n+1\right\rangle \qquad a\left|n\right\rangle \rightarrow \left|n-1\right\rangle$$

- This means that the first operator shifts the SHO to the next higher eigenstate, while the second one shifts the SHO to the previous lower eigenstate!

- *Question: what is the result of the action of the operator* $N = a^{\dagger}a$ *on a SHO eigenstate?*

…the algebraic solution-c…

- We define the raising and lowering operators as follows:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right), \qquad a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

where *x* and *p* are the position and momentum operators. With the help of these operators the Hamiltonian takes the form

$$H = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right) \qquad N = a^{\dagger}a$$

Question 1: Are the creation and annihilation operators Hermitian?

Question 2: Express the SHO Hamiltonian as a "function" of the raising and lowering operators.

…the algebraic solution-d…

Prove the following relations ($N = a^\dagger a$)

$$\left[a,\ a^\dagger\right] = 1, \qquad \left[N,\ a\right] = -a, \qquad \left[N,\ a^\dagger\right] = a^\dagger$$

- The most characteristic property of the energy spectrum of the SHO is the equal distance between successive energy eigenvalues.
- *Question: Prove the characteristic property of the SHO energy spectrum property. Find the energy eigenvalues of SHO.*

…the algebraic solution-e…

- The raising and lowering operators are known as creation and destruction operators respectively since the first "creates" a quantum of energy $\hbar\omega$ and thus raises the SHO to the next higher state, while the second "destroys" a quantum of energy $\hbar\omega$ and thus brings the SHO down to the next lower state.
- The operator $N = a^\dagger a$ is known a number operator since it gives the number of energy quanta in a state $|n\rangle$

…the algebraic solution-f…

- We will prove that the proper forms of the raising and lowering operators are:

$$a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \qquad a^{\dagger}\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$$

- With them one can built any state $\left|n\right\rangle$

$$\left|n\right\rangle = \frac{1}{\sqrt{n!}}\left(a^{\dagger}\right)^{n}\left|0\right\rangle$$

from the vacuum state $\left|0\right\rangle$ (n=0) .

…an interesting theorem…

- If for an operator *H* we can find an operator *A* for which, $\left[H, A\right] = \xi A$ then: a) Operator *H* has equidistant eigenvalues

  b) If $\xi < 0$, operator *A* is a lowering operator, while if $\xi > 0$, operator *A* is a raising operator

Further Reading

- https://ocw.mit.edu/courses/physics/8-04-quantum-physics-i-spring-2013/lecture-videos/lecture-9/