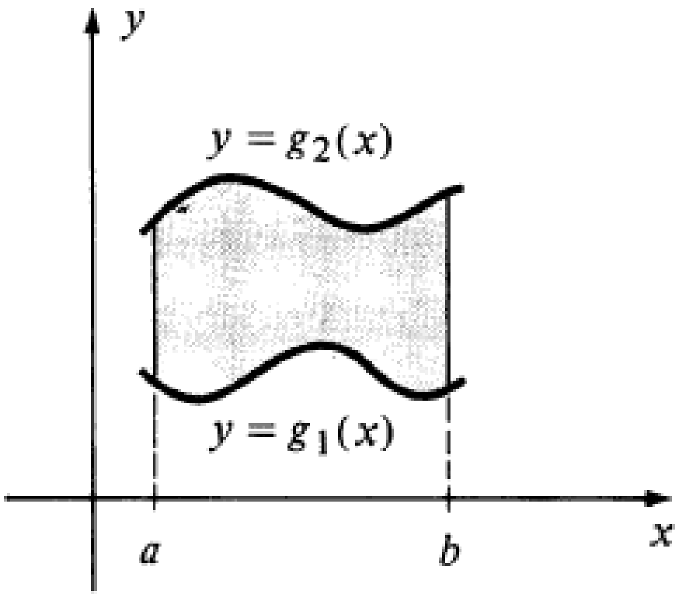
Faculty of Engineering
Mechanical Engineering Department

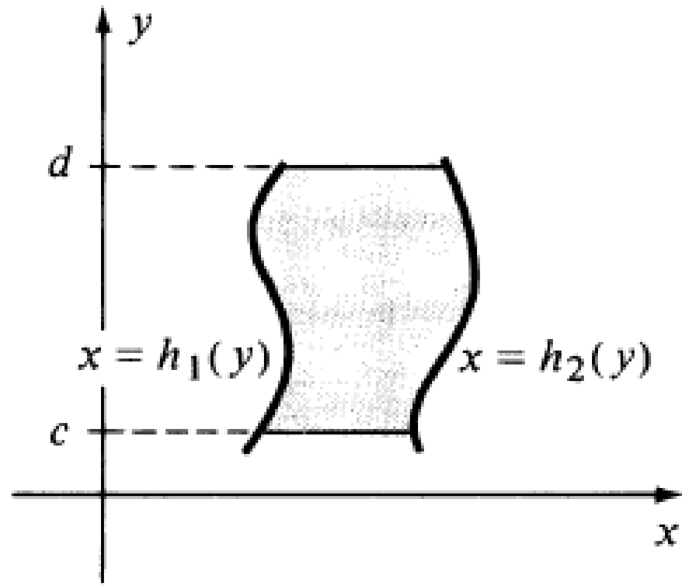
CALCULUS FOR ENGINEERS
MATH 1110

Instructor:
Dr. Mohamed El-Shazly
Assistant Prof. of Mechanical Design and Tribology
mohamed.elshazly@ams-sae.com
Office: S053

DOUBLE INTEGRALS



(i) Region of Type I



(ii) Region of Type II

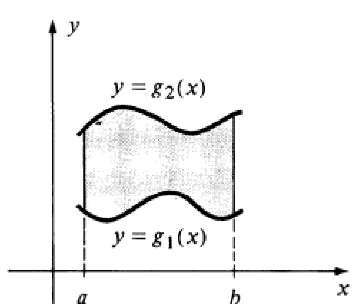
Theorem (17.9)

(i) Let R be a region of Type I that lies between the graphs of $y = g_1(x)$ and $y = g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$. If f is continuous on R, then

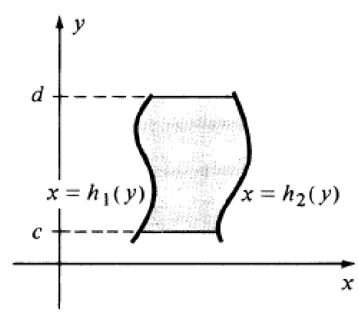
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

(ii) Let R be a region of Type II that lies between the graphs of $x = h_1(y)$ and $x = h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$. If f is continuous on R then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$



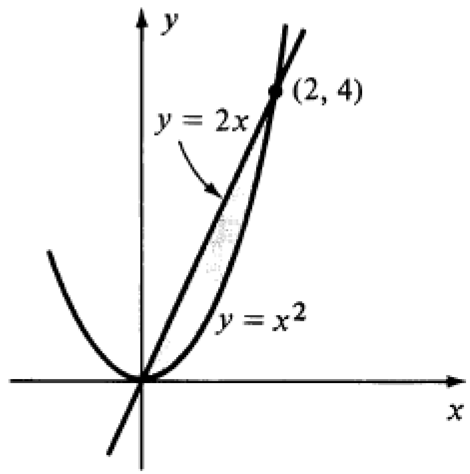
(i) Region of Type I



(ii) Region of Type II

Example 1

Evaluate $\iint_R (x^3 + 4y)\, dA$, where R is the region in the xy-plane bounded by the graphs of the equations $y = x^2$ and $y = 2x$.



$$\iint_R (x^3 + 4y)\, dA = \int_0^2 \int_{x^2}^{2x} (x^3 + 4y)\, dy\, dx.$$

Solution By (i) of Definition (17.8) the integral equals

$$\begin{aligned}
\int_0^2 \left[\int_{x^2}^{2x} (x^3 + 4y)\, dy \right] dx &= \int_0^2 \Big[x^3 y + 2y^2 \Big]_{x^2}^{2x} dx \\
&= \int_0^2 [(2x^4 + 8x^2) - (x^5 + 2x^4)]\, dx \\
&= \frac{8}{3}x^3 - \frac{1}{6}x^6 \Big]_0^2 = \frac{32}{3}.
\end{aligned}$$

If we regard R as a region of Type II, then the left-hand boundary is the graph of $x = \frac{1}{2}y$ and the right-hand boundary is the graph of $x = \sqrt{y}$, where $0 \leq y \leq 4$. Hence by (ii) of Theorem 17.9,

$$\iint_R f(x, y)\, dA = \int_0^4 \int_{(1/2)y}^{\sqrt{y}} (x^3 + 4y)\, dx\, dy$$

$$= \int_0^4 \left[\frac{1}{4}x^4 + 4yx\right]_{(1/2)y}^{\sqrt{y}} dy$$

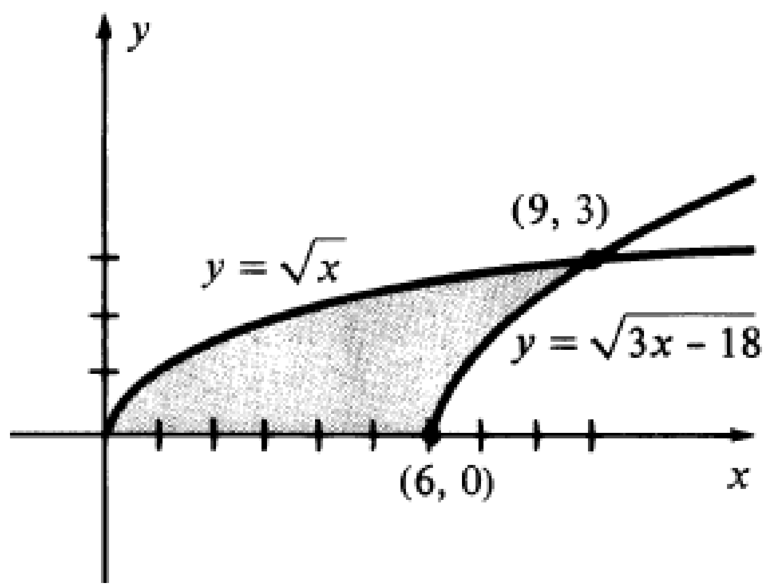
$$= \int_0^4 \left[\left(\frac{1}{4}y^2 + 4y^{3/2}\right) - \left(\frac{1}{64}y^4 + 2y^2\right)\right] dy = \frac{32}{3}.$$

Example 2

Let R be the region bounded by the graphs of the equations $y = \sqrt{x}$, $y = \sqrt{3x - 18}$ and $y = 0$. If f is continuous on R, express the double integral $\iint_R f(x, y)\, dA$ in terms of iterated integrals by

(a) using only part (i) of Theorem 17.9.

(b) using only part (ii) of Theorem 17.9.



(a) If we wish to use only part (i) of Theorem 17.9, then it is necessary to employ two iterated integrals, because if $0 \leq x \leq 6$, then the lower boundary of the region is the graph of $y = 0$, whereas if $6 \leq x \leq 9$, the lower boundary is the graph of $y = \sqrt{3x - 18}$. If R_1 denotes the part of the region R that lies between $x = 0$ and $x = 6$, and if R_2 denotes the part between $x = 6$ and $x = 9$, then both R_1 and R_2 are regions of Type I. Hence

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

$$= \int_0^6 \int_0^{\sqrt{x}} f(x, y)\, dy\, dx + \int_6^9 \int_{\sqrt{3x-18}}^{\sqrt{x}} f(x, y)\, dy\, dx.$$

(b) To use (ii) of Theorem 17.9 we must solve each of the given equations for x in terms of y, obtaining

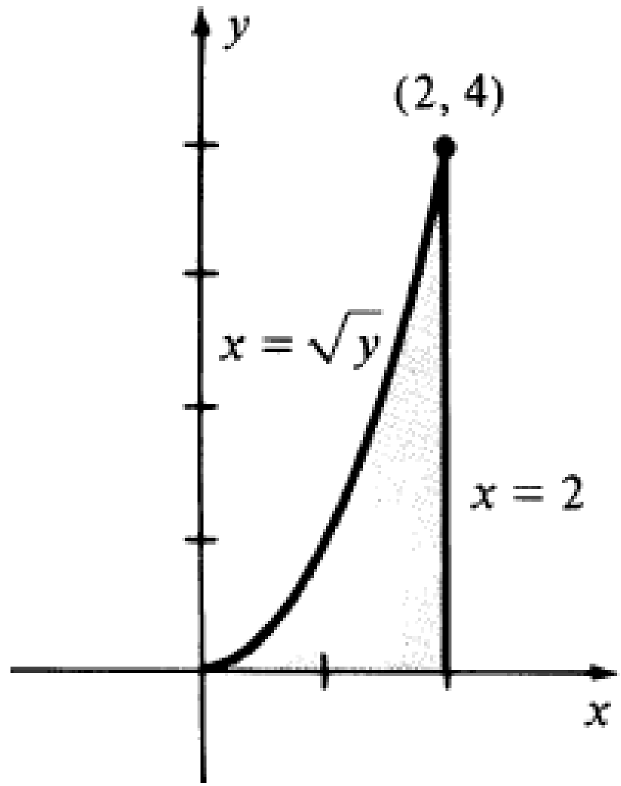
$$x = y^2 \quad \text{and} \quad x = \frac{y^2 + 18}{3} = \frac{1}{3}y^2 + 6.$$

Only one iterated integral is required in this case since R is a region of Type II. Thus

$$\iint_R f(x, y)\, dA = \int_0^3 \int_{y^2}^{(1/3)y^2 + 6} f(x, y)\, dx\, dy. \qquad \blacksquare$$

Example 3

Given $\int_0^4 \int_{\sqrt{y}}^2 y \cos x^5 \, dx \, dy$, reverse the order of integration and evaluate the resulting integral.



Solution Since the given order of integration is $dx\,dy$, the region R is of Type II. As illustrated in Figure 17.10, the left-hand and right-hand boundaries are the graphs of $x = \sqrt{y}$ and $x = 2$, respectively, and $0 \le y \le 4$.

Note that R is also a region of Type I whose lower and upper boundaries are given by $y = 0$ and $y = x^2$, respectively, and where $0 \le x \le 2$. Hence by Theorem 17.9,

$$\int_0^4 \int_{\sqrt{y}}^2 y \cos x^5 \, dx \, dy = \iint_R y \cos x^5 \, dA = \int_0^2 \int_0^{x^2} y \cos x^5 \, dy \, dx$$

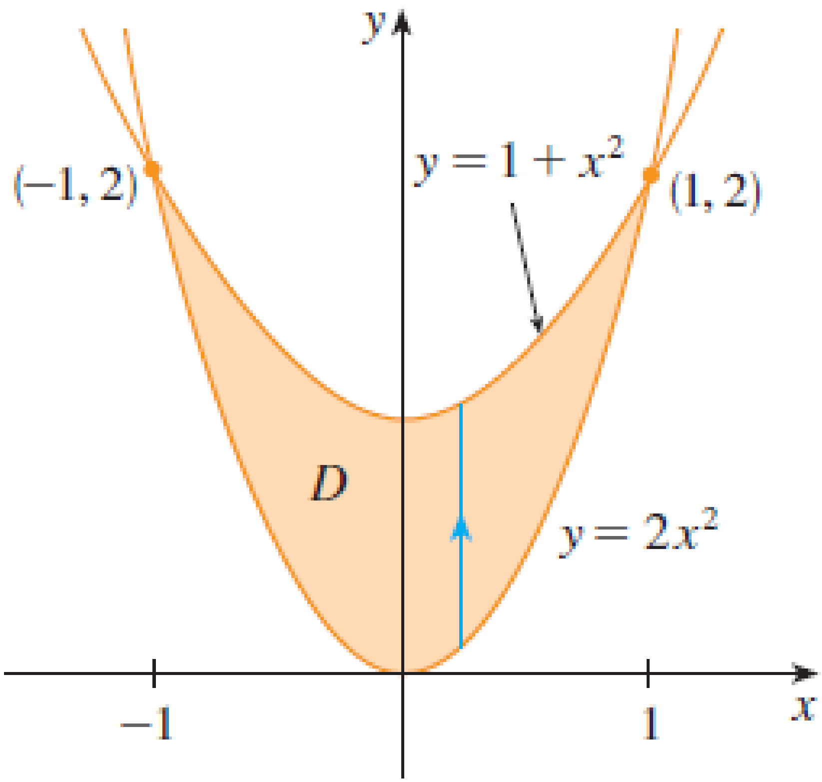
$$= \int_0^2 \frac{y^2}{2} \cos x^5 \Big]_0^{x^2} dx = \int_0^2 \frac{x^4}{2} \cos x^5 \, dx$$

$$= \frac{1}{10} \int_0^2 \cos x^5 (5x^4) \, dx$$

$$= \frac{1}{10} \sin x^5 \Big]_0^2 = \frac{1}{10} \sin 32 \approx 0.055.$$

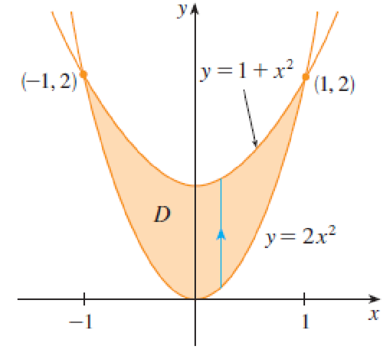
Example 4

Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



SOLUTION The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region D, sketched in Figure 8, is a type I region but not a type II region and we can write

$$D = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1 + x^2\}$$



Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \Big[xy + y^2\Big]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \Big]_{-1}^{1} = \frac{32}{15}$$

Example 5

Evaluate $I = \iint_{\mathcal{R}} x\,dA$, where $\mathcal{R}$ is the region bounded by $y = x$ and $y = x^2$.

The curves $y = x$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$

and, for $0 < x < 1$, $y = x$ is above $y = x^2$

$$I = \int_0^1 \int_{x^2}^{x} x\,dy\,dx = \int_0^1 xy\,\Big]_{x^2}^{x} dx$$

$$= \int_0^1 (x^2 - x^3)\,dx = \left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right)\Big]_0^1 = \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{1}{12}$$