



Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS

MATH 1110

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Areas and Volumes

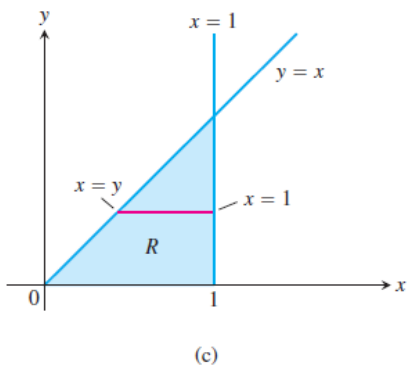
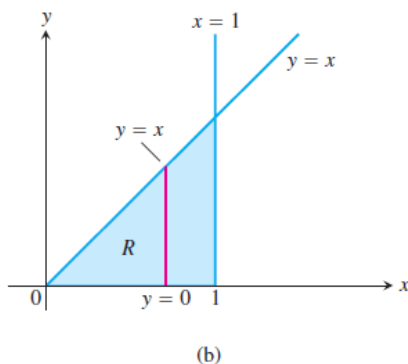
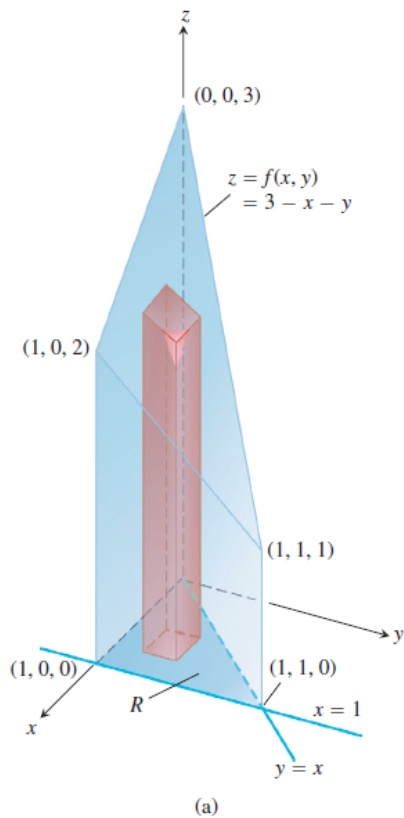
Volume Equals to:

$$\text{Volume} = \iint_R f(x, y) \, dA,$$

Example 1:

Find the volume of the prism whose base is the triangle in the xy -plane bounded by the x -axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$



Solution1:

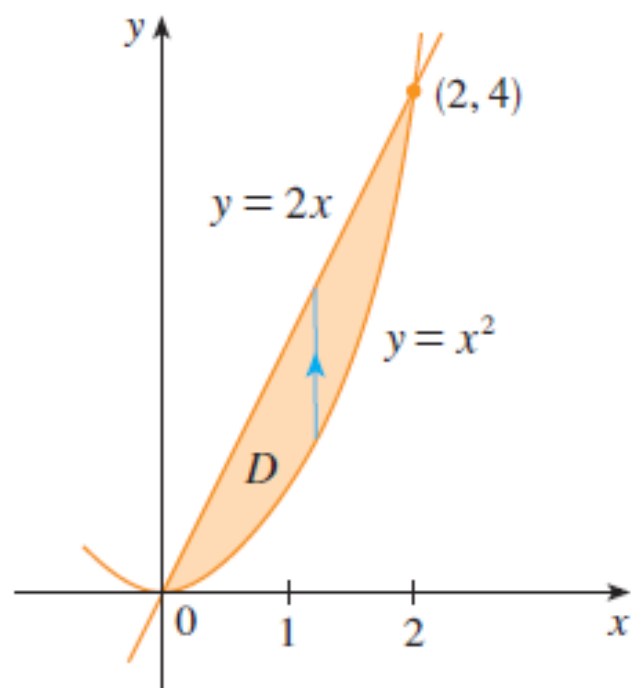
$$\begin{aligned} V &= \int_0^1 \int_0^x (3 - x - y) dy dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx \\ &= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1. \end{aligned}$$

When the order of integration is reversed (Figure 15.11c), the integral for the volume is

$$\begin{aligned} V &= \int_0^1 \int_y^1 (3 - x - y) dx dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy \\ &= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy \\ &= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1. \end{aligned}$$

The two integrals are equal, as they should be.

EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy -plane bounded by the line $y = 2x$ and the parabola $y = x^2$.



Solution 2:

$$V = \iint_D (x^2 + y^2) dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) dy dx$$

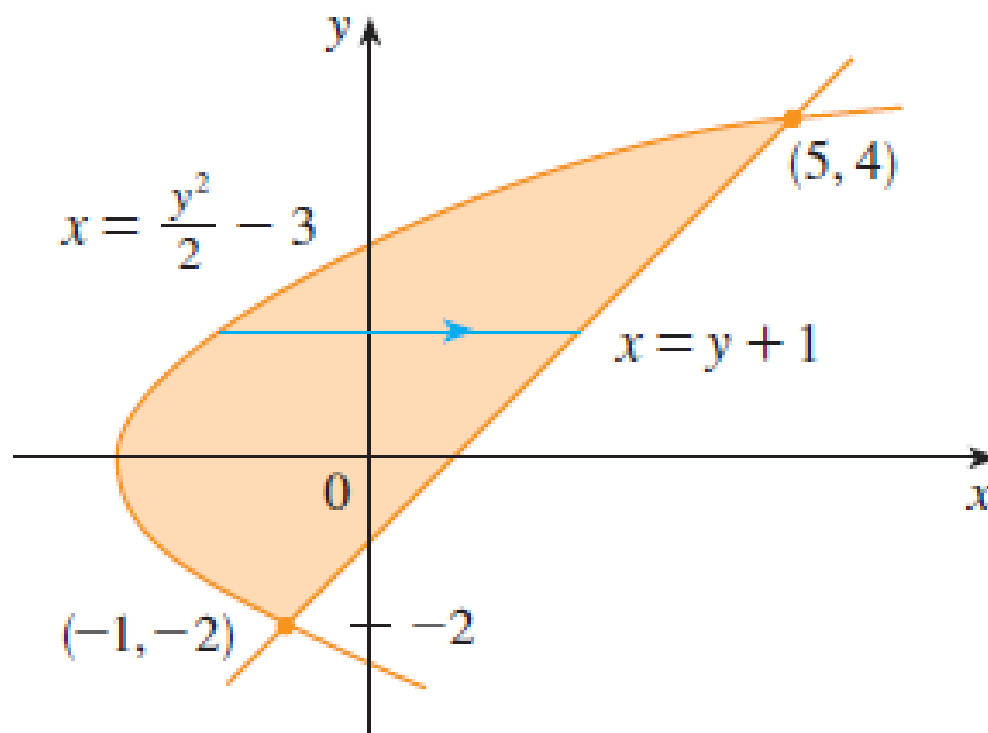
$$= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx$$

$$= \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2 x^2 - \frac{(x^2)^3}{3} \right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx$$

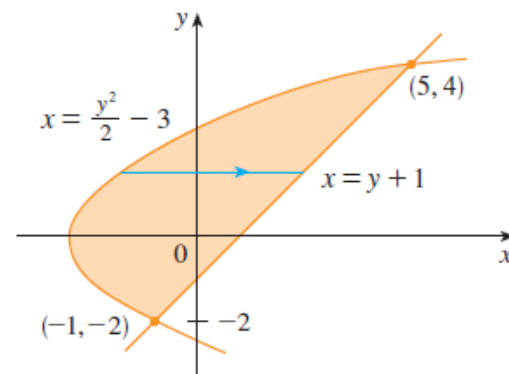
$$= -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \Big|_0^2 = \frac{216}{35}$$

EXAMPLE 3 Evaluate $\iint_D xy \, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

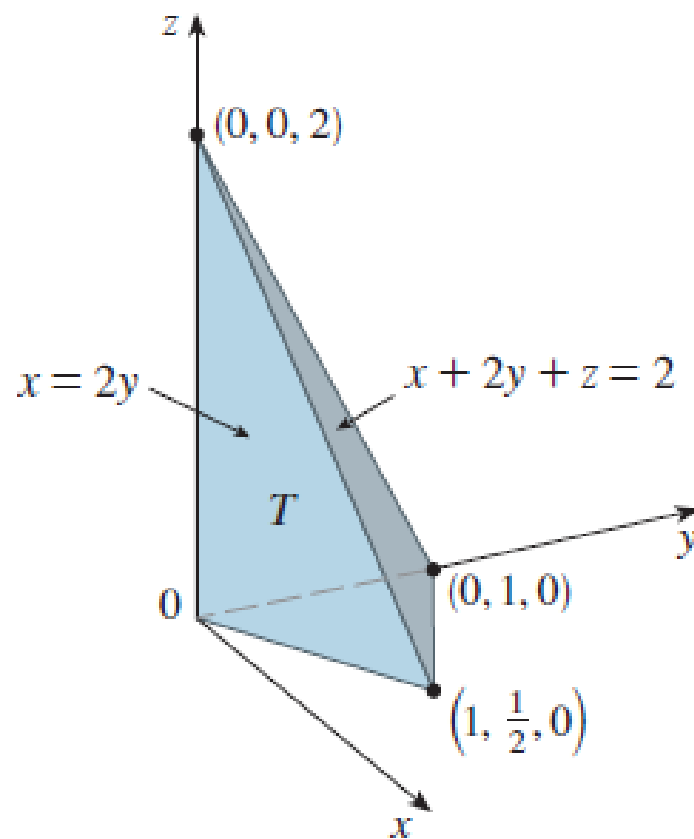


Solution 3:

$$\begin{aligned}\iint_D xy \, dA &= \int_{-2}^4 \int_{\frac{1}{2}y^2-3}^{y+1} xy \, dx \, dy = \int_{-2}^4 \left[\frac{x^2}{2} y \right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy \\&= \frac{1}{2} \int_{-2}^4 y \left[(y+1)^2 - \left(\frac{1}{2}y^2 - 3 \right)^2 \right] dy \\&= \frac{1}{2} \int_{-2}^4 \left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y \right) dy \\&= \frac{1}{2} \left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2 \right]_{-2}^4 = 36\end{aligned}$$



EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.



Solution 4:

Therefore

$$\begin{aligned}
 V &= \iint_D (2 - x - 2y) \, dA \\
 &= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y) \, dy \, dx \\
 &= \int_0^1 \left[2y - xy - y^2 \right]_{y=x/2}^{y=1-x/2} dx \\
 &= \int_0^1 \left[2 - x - x \left(1 - \frac{x}{2} \right) - \left(1 - \frac{x}{2} \right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4} \right] dx \\
 &= \int_0^1 (x^2 - 2x + 1) \, dx = \left[\frac{x^3}{3} - x^2 + x \right]_0^1 = \frac{1}{3}
 \end{aligned}$$

