Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS
MATH 1110

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Moment of Inertia

"second moment"

The Area Moment Of Inertia of a beams cross-sectional area measures the beams ability to resist bending. The larger the Moment of Inertia the less the beam will bend.

$$I_x = \iint_D y^2 \rho(x, y)\, dA$$

$$I_y = \iint_D x^2 \rho(x, y)\, dA$$

Polar Moment of Inertia

The moment of inertia about the origin

$$I_0 = \iint_D (x^2 + y^2)\, \rho(x, y)\, dA$$

$$I_0 = I_x + I_y.$$

Example 1

Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center the origin, and radius a.

SOLUTION The boundary of D is the circle $x^2 + y^2 = a^2$ and in polar coordinates D is described by $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant r \leqslant a$. Let's compute I_0 first:

$$I_0 = \iint_D (x^2 + y^2)\rho \, dA = \rho \int_0^{2\pi} \int_0^a r^2 \, r \, dr \, d\theta$$

$$= \rho \int_0^{2\pi} d\theta \int_0^a r^3 \, dr = 2\pi\rho \left[\frac{r^4}{4} \right]_0^a = \frac{\pi\rho a^4}{2}$$

Instead of computing I_x and I_y directly, we use the facts that $I_x + I_y = I_0$ and $I_x = I_y$ (from the symmetry of the problem). Thus

$$I_x = I_y = \frac{I_0}{2} = \frac{\pi\rho a^4}{4}$$

Example 2

Find the moments of inertia of the triangle bounded by $3x + 4y = 24$, $x = 0$, and $y = 0$, and having density 1.

Solution 2

The moment of inertia with respect to the x-axis is $I_x = \int_0^8 \int_0^{6-(3/4)x} y^2\, dy\, dx =$

$$\int_0^8 \frac{1}{3} y^3 \Big]_0^{6-(3/4)x} dx =$$

$$\frac{9}{64}\int_0^8 (8-x)^3\, dx = \frac{9}{64}(-1)\frac{(8-x)^4}{4}\Big]_0^8 = \frac{9}{64}\cdot\frac{8^4}{4} = 144.$$

The moment of inertia with respect to the y-axis is

$$I_y = \int_0^8 \int_0^{6-(3/4)x} x^2 \, dy \, dx =$$

$$\int_0^8 x^2 \cdot \frac{3}{4}(8 - x) \, dx = \frac{3}{4}\left(\frac{8}{3}x^3 - \frac{x^4}{4}\right)\Big]_0^8 =$$

$$\frac{3}{4} \cdot \frac{1}{12}(8)^4 = 256.$$

Example 3

Find the moments of inertia for the lamina of:

D is bounded by the parabola $x = y^2$ and the line $y = x - 2$;
$\rho(x, y) = 3$

Solution 3

$$I_x = \int_{-1}^{2} \int_{y^2}^{y+2} 3y^2 \, dx\, dy = \int_{-1}^{2} (3y^3 + 6y^2 - 3y^4)\, dy$$

$$= \left[\frac{3}{4} y^4 + 2y^3 - \frac{3}{5} y^5 \right]_{-1}^{2} = \frac{189}{20}$$

$$I_y = \int_{-1}^{2}\int_{y^2}^{y+2} 3x^2\,dx\,dy = \int_{-1}^{2}\left[(y+2)^3 - y^6\right]dy$$

$$= \left[\frac{1}{4}(y+2)^4 - \frac{1}{7}y^7\right]_{-1}^{2} = \frac{1269}{28}$$, and $I_0 = I_x + I_y = \frac{1917}{35}$