



Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS

MATH 1110

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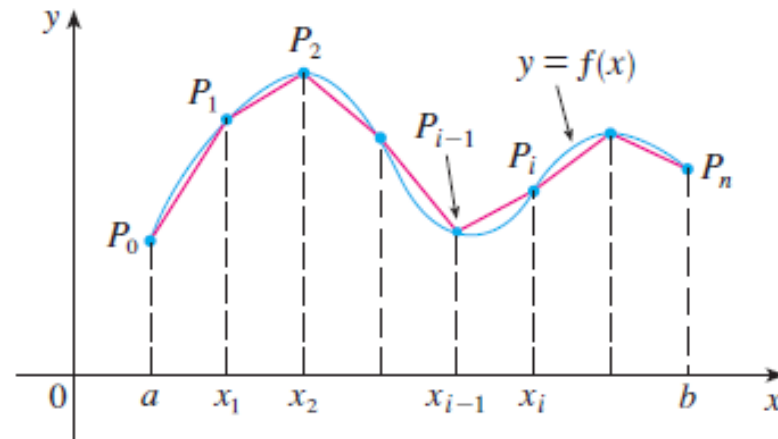
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Line Integral

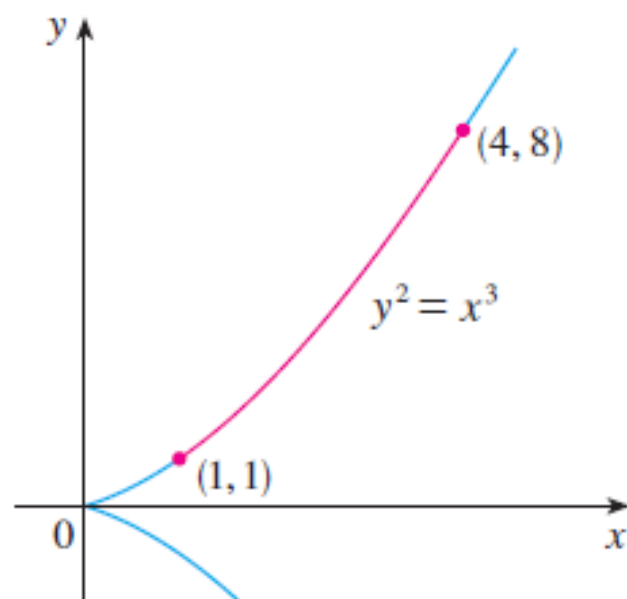
Arc Length



$$L = \lim_{n \rightarrow \infty} \sum_{l=1}^n |P_{l-1}P_l|$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

EXAMPLE 1 Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.



SOLUTION For the top half of the curve we have

$$y = x^{3/2} \qquad \frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

and so the arc length formula gives

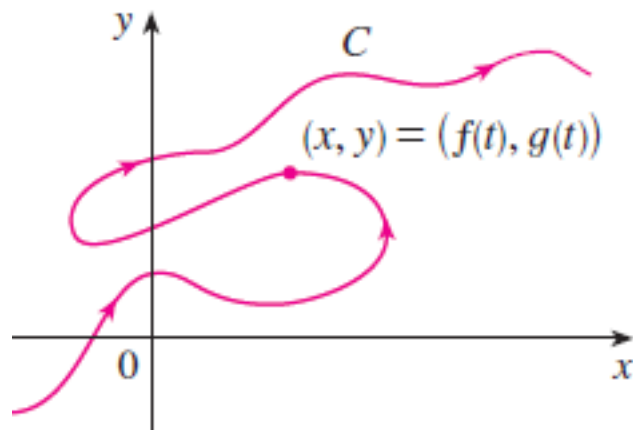
$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_1^4 \sqrt{1 + \frac{9}{4}x} dx$$

If we substitute $u = 1 + \frac{9}{4}x$, then $du = \frac{9}{4} dx$. When $x = 1$, $u = \frac{13}{4}$; when $x = 4$, $u = 10$.

Therefore

$$\begin{aligned} L &= \frac{4}{9} \int_{13/4}^{10} \sqrt{u} du = \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \Big|_{13/4}^{10} \\ &= \frac{8}{27} \left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2} \right] = \frac{1}{27} (80\sqrt{10} - 13\sqrt{13}) \end{aligned}$$

Curves Defined by Parametric Equations



- Imagine that a particle moves along the curve C shown in Figure 1. It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test.
- But the x - and y -coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$.
- Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.
- Suppose that x and y are both given as functions of a third variable (called a **parameter**) by the equation

$$x = f(t) \quad y = g(t)$$

Calculus with Parametric Curve

Arc Length

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Using Parametric equation $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $dx/dt = f'(t) > 0$.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \frac{dx}{dt} dt$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example 2

Find the Length of one arch of the cycloid $x = r (\theta - \sin\theta)$, $y = r (1 - \cos\theta)$

Soultion

$$\frac{dx}{d\theta} = r(1 - \cos\theta) \quad \text{and} \quad \frac{dy}{d\theta} = r \sin\theta$$

we have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1 - \cos\theta)^2 + r^2 \sin^2\theta} d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)} d\theta \\ &= r \int_0^{2\pi} \sqrt{2(1 - \cos\theta)} d\theta \end{aligned}$$

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives $1 - \cos \theta = 2 \sin^2(\theta/2)$. Since $0 \leq \theta \leq 2\pi$, we have $0 \leq \theta/2 \leq \pi$ and so $\sin(\theta/2) \geq 0$. Therefore

$$\sqrt{2(1 - \cos \theta)} = \sqrt{4 \sin^2(\theta/2)} = 2 |\sin(\theta/2)| = 2 \sin(\theta/2)$$

and so

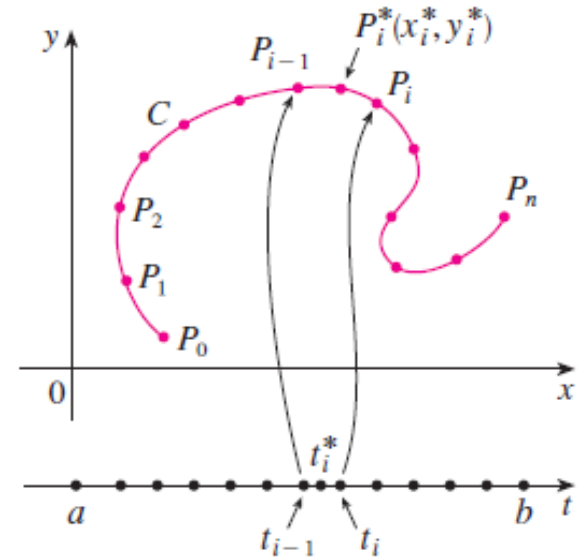
$$\begin{aligned} L &= 2r \int_0^{2\pi} \sin(\theta/2) d\theta = 2r [-2 \cos(\theta/2)]_0^{2\pi} \\ &= 2r[2 + 2] = 8r \end{aligned}$$

Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a,b]$, we integrate over a curve C .

Such integrals are called *line integrals*, although “curve integrals” would be better terminology.

They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.



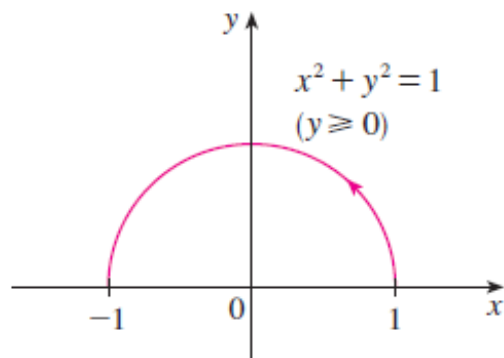
$$\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

EXAMPLE 1 Evaluate $\int_C (2 + x^2 y) \, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \quad y = \sin t$$

and the upper half of the circle is described by the parameter interval $0 \leq t \leq \pi$. (See Figure 3.) Therefore Formula 3 gives



$$\begin{aligned} \int_C (2 + x^2 y) \, ds &= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t} \, dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t) \, dt = \left[2t - \frac{\cos^3 t}{3} \right]_0^\pi \\ &= 2\pi + \frac{2}{3} \end{aligned}$$

EXAMPLE 2 Evaluate $\int_C 2x \, ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.

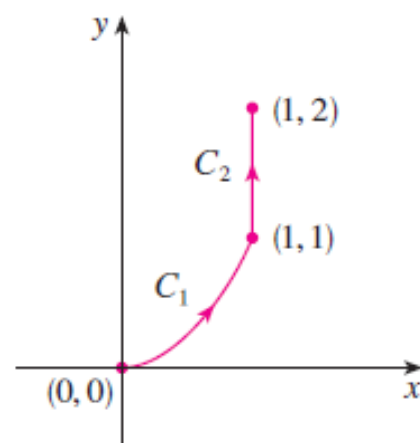


FIGURE 5

SOLUTION The curve C is shown in Figure 5. C_1 is the graph of a function of x , so we can choose x as the parameter and the equations for C_1 become

$$x = x \quad y = x^2 \quad 0 \leq x \leq 1$$

Therefore

$$\begin{aligned} \int_{C_1} 2x \, ds &= \int_0^1 2x \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^1 2x \sqrt{1 + 4x^2} \, dx \\ &= \frac{1}{4} \cdot \frac{2}{3} (1 + 4x^2)^{3/2} \Big|_0^1 = \frac{5\sqrt{5} - 1}{6} \end{aligned}$$

On C_2 we choose y as the parameter, so the equations of C_2 are

$$x = 1 \quad y = y \quad 1 \leq y \leq 2$$

and

$$\int_{C_2} 2x \, ds = \int_1^2 2(1) \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2} \, dy = \int_1^2 2 \, dy = 2$$

Thus

$$\int_C 2x \, ds = \int_{C_1} 2x \, ds + \int_{C_2} 2x \, ds = \frac{5\sqrt{5} - 1}{6} + 2$$



Example 2

Evaluate $\int_C xy^4 ds$ where C is the right half of the circle, $x^2 + y^2 = 16$ rotated in the counter clockwise direction.

Solution

We first need a parameterization of the circle. This is given by,

$$x = 4 \cos t \quad y = 4 \sin t$$

We now need a range of t 's that will give the right half of the circle. The following range of t 's will do this.

$$-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

Now, we need the derivatives of the parametric equations and let's compute ds .

$$\frac{dx}{dt} = -4 \sin t \quad \frac{dy}{dt} = 4 \cos t$$

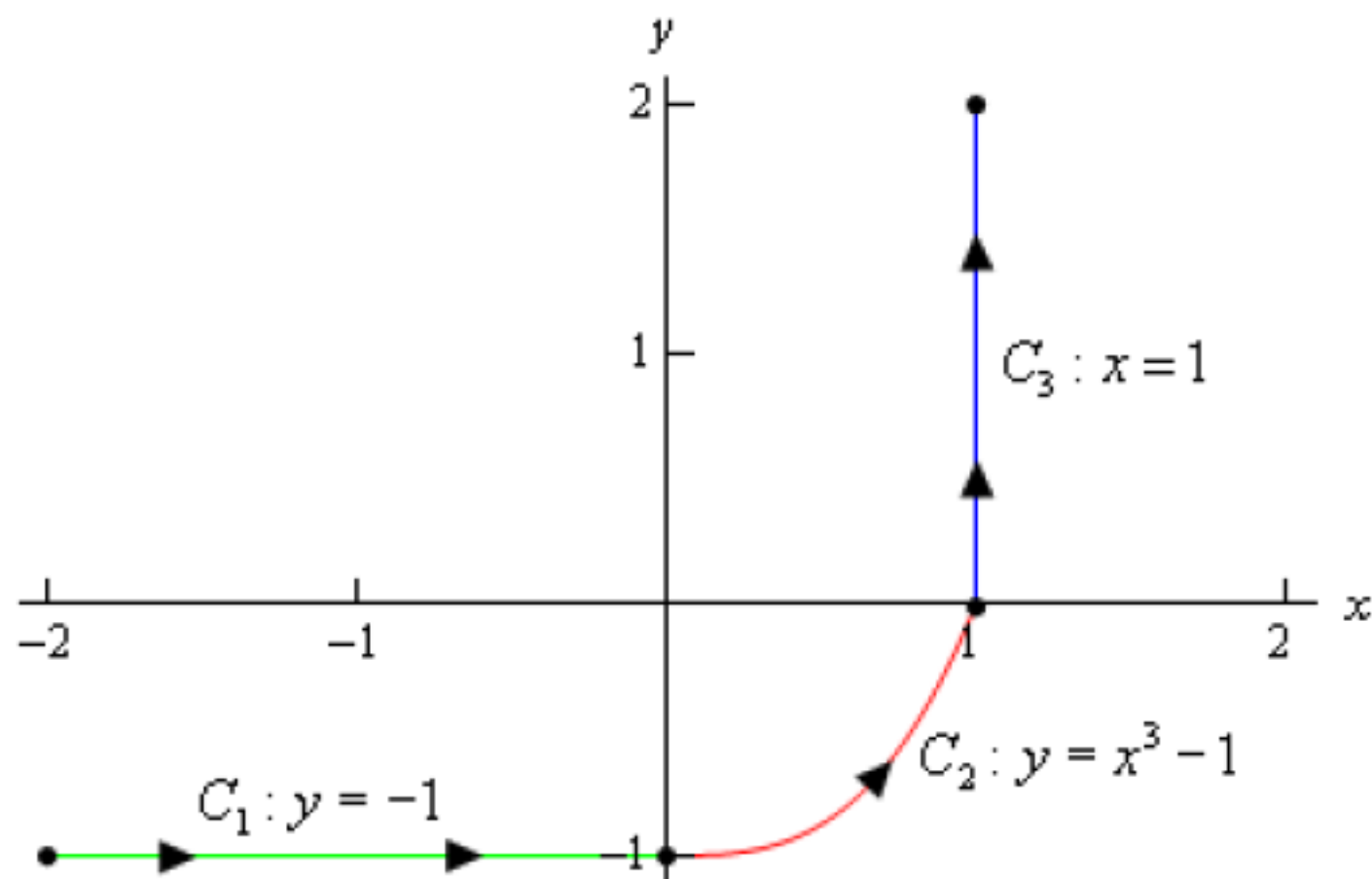
$$ds = \sqrt{16 \sin^2 t + 16 \cos^2 t} dt = 4 dt$$

The line integral is then,

$$\begin{aligned}\int_C xy^4 ds &= \int_{-\pi/2}^{\pi/2} 4 \cos t (4 \sin t)^4 (4) dt \\&= 4096 \int_{-\pi/2}^{\pi/2} \cos t \sin^4 t dt \\&= \frac{4096}{5} \sin^5 t \bigg|_{-\pi/2}^{\pi/2} \\&= \frac{8192}{5}\end{aligned}$$

Example 3

Evaluate $\int_C 4x^3 ds$ where C is the curve shown below.



$$C_1 : x = t, y = -1, \quad -2 \leq t \leq 0$$

$$C_2 : x = t, y = t^3 - 1, \quad 0 \leq t \leq 1$$

$$C_3 : x = 1, y = t, \quad 0 \leq t \leq 2$$

Now let's do the line integral over each of these curves.

$$\int_{C_1} 4x^3 ds = \int_{-2}^0 4t^3 \sqrt{(1)^2 + (0)^2} dt = \int_{-2}^0 4t^3 dt = t^4 \Big|_{-2}^0 = -16$$

$$\begin{aligned} \int_{C_2} 4x^3 ds &= \int_0^1 4t^3 \sqrt{(1)^2 + (3t^2)^2} dt \\ &= \int_0^1 4t^3 \sqrt{1 + 9t^4} dt \\ &= \frac{1}{9} \left(\frac{2}{3} \right) (1 + 9t^4)^{\frac{3}{2}} \Big|_0^1 = \frac{2}{27} \left(10^{\frac{3}{2}} - 1 \right) = 2.268 \end{aligned}$$

$$\int_{C_3} 4x^3 ds = \int_0^2 4(1)^3 \sqrt{(0)^2 + (1)^2} dt = \int_0^2 4 dt = 8$$

Finally, the line integral that we were asked to compute is,

$$\begin{aligned} \int_C 4x^3 ds &= \int_{C_1} 4x^3 ds + \int_{C_2} 4x^3 ds + \int_{C_3} 4x^3 ds \\ &= -16 + 2.268 + 8 \\ &= -5.732 \end{aligned}$$