Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS
MATH 1110

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Surface Integrals

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Evaluating a Surface Integral

Let S be a surface with equation $z = g(x, y)$ and let R be its projection onto the xy-plane. If g, g_x, and g_y are continuous on R and f is continuous on S, then the surface integral of f over S is

$$\int_S\int f(x, y, z)\, dS = \int_R\int f(x, y, g(x, y))\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\, dA.$$

Example 1 – *Evaluating a Surface Integral*

Evaluate the surface integral

$$\int_S\int (y^2 + 2yz)\, dS$$

where S is the first-octant portion of the plane

$2x + y + 2z = 6$.

Solution:

Begin by writing S as

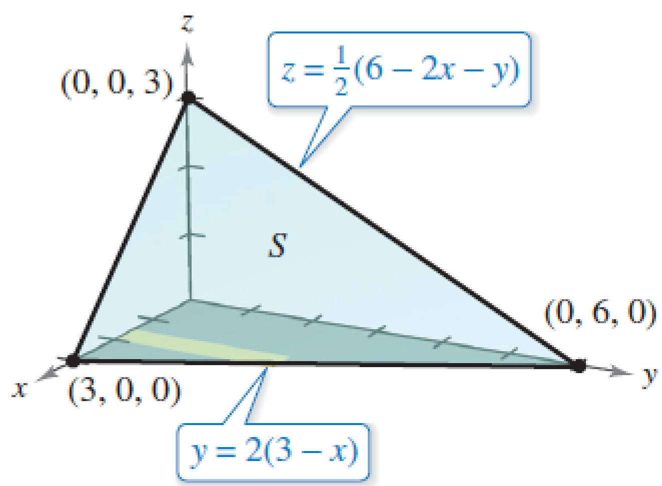
$$z = \frac{1}{2}(6 - 2x - y)$$

$$g(x, y) = \frac{1}{2}(6 - 2x - y).$$

Example 1 – *Solution*

Using the partial derivatives $g_x(x, y) = -1$ and $g_y(x, y) = -\frac{1}{2}$, you can write

$$\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2} = \sqrt{1 + 1 + \frac{1}{4}} = \frac{3}{2}.$$



Example 1 – *Solution*

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Using Figure 15.45 and Theorem 15.10, you obtain

$$\int_S\int (y^2 + 2yz)\, dS = \int_R\int f(x, y, g(x, y))\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\, dA$$

$$= \int_R\int \left[y^2 + 2y\left(\frac{1}{2}\right)(6 - 2x - y)\right]\left(\frac{3}{2}\right) dA$$

$$= 3\int_0^3\int_0^{2(3-x)} y(3 - x)\, dy\, dx$$

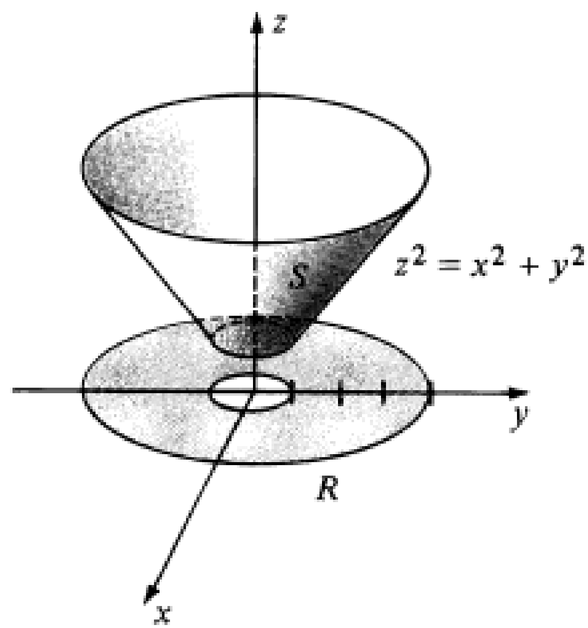
$$= 6\int_0^3 (3 - x)^3\, dx$$

$$= -\frac{3}{2}(3 - x)^4\Big]_0^3$$

$$= \frac{243}{2}.$$

Example 2

Evaluate $\iint_S x^2 z \, dS$ where S is the portion of the circular cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 4$.



Example 2: Solution

Solution As shown in Figure 18.16, the projection R of S onto the xy-plane is the annular region bounded by circles of radii 1 and 4 with centers at the origin.

If we write the equation for S in the form

$$z = (x^2 + y^2)^{1/2} = f(x, y)$$

then

$$f_x(x, y) = \frac{x}{(x^2 + y^2)^{1/2}} \quad \text{and} \quad f_y(x, y) = \frac{y}{(x^2 + y^2)^{1/2}}.$$

Applying (18.21) and noting that the radical reduces to $\sqrt{2}$, we obtain

$$\iint_S x^2 z\, dS = \iint_R x^2(x^2 + y^2)^{1/2}\sqrt{2}\, dx\, dy.$$

Using polar coordinates to evaluate the double integral,

$$\iint_S x^2 z\, dS = \int_0^{2\pi}\int_1^4 (r^2\cos^2\theta) r\sqrt{2} r\, dr\, d\theta = \sqrt{2}\int_0^{2\pi} \cos^2\theta \left.\frac{r^5}{5}\right]_1^4 d\theta$$

$$= \frac{1023\sqrt{2}}{5}\int_0^{2\pi} \frac{1 + \cos 2\theta}{2}\, d\theta = \frac{1023\sqrt{2}}{10}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{1023\sqrt{2}\pi}{5} \approx 909.0.$$

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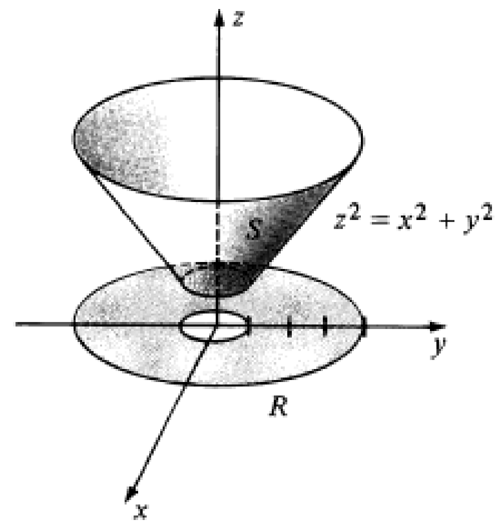

Derivatives Involving Powers of Functions

Example 2 Find $f'(x)$ if $f(x) = \sqrt[3]{5x^2 - x + 4}$.

Solution Writing $f(x) = (5x^2 - x + 4)^{1/3}$ and using the Power Rule for Functions with $n = 1/3$,

$$\begin{aligned} f'(x) &= \tfrac{1}{3}(5x^2 - x + 4)^{-2/3} D_x(5x^2 - x + 4) \\ &= \left(\frac{1}{3}\right) \frac{1}{(5x^2 - x + 4)^{2/3}} (10x - 1) \\ &= \frac{10x - 1}{3\sqrt[3]{(5x^2 - x + 4)^2}}. \end{aligned}$$

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