PHYS-505/551

Angular Momentum & Rotations

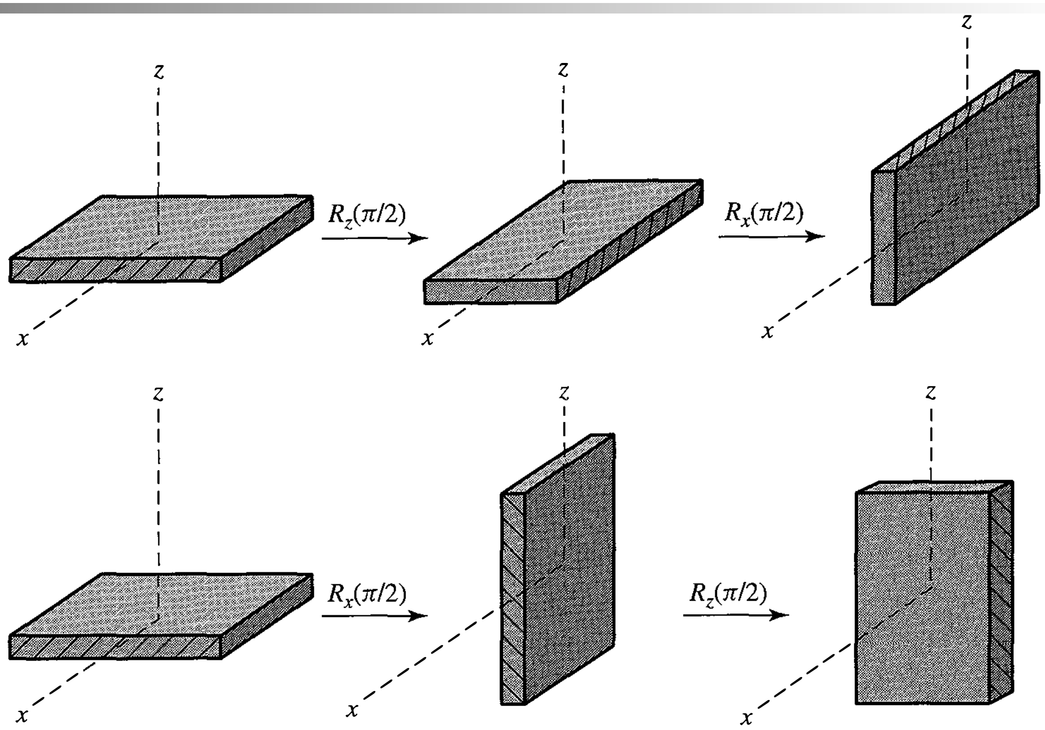
Lecture-4

Introduction-a

- From elementary classical physics we know that rotations about the same axis commute, whereas rotations about different axes do not.
- This means in simple words: A rotation by 30^0 around z-axis followed by a rotation by 60^0 degrees again around z-axis is the same as a rotation by 60^0 around z-axis followed by a rotation by 30^0 degrees again around z-axis.
- We cannot tell the same for a rotation around z-axis followed by a rotation around, say x-axis.

Introduction-b

- We cannot tell the same for a rotation around z-axis followed by a rotation around, say x-axis.



Rotation of vectors-a

- Consider a vector $\mathbf{V} = (V_x, V_y, V_z)$. When we rotate it, its components change. The new and the old components are related via an orthogonal 3×3 matrix R as follows

$$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = \begin{pmatrix} & & \\ & R & \\ & & \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$RR^T = R^T R = 1$$

Rotation of vectors-b

- In here we consider rotation of a physical system **not** of the system of axes (*active rotation*).
- For a rotation around z-axis angle ϕ is taken as positive when the rotation in question is counterclockwise in the xy-plane **as viewed from the positive z-side.**
- With the above conventions we have the matrix $R_z(\phi)$ for the rotations around z-axis

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If we considered rotation of the axes then the above matrix would represent a *clockwise* rotation when viewed from the positive z-side (*passive rotation*).

Rotation of vectors-c

- By cyclic permutation of x, y, z, i.e. $x \rightarrow y$, $y \rightarrow z$, $z \rightarrow x$ we get the matrices for rotations around x, and y axes by an angle ϕ.

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \quad R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

Infinitesimal rotation of vectors-a

- If we consider very small angles of rotation (call them ε) then we know that approximately: $\cos\phi = 1 - \frac{\varepsilon^2}{2}$ $\sin\phi = \varepsilon$

$$R_z(\phi) = \begin{pmatrix} 1-\varepsilon^2/2 & -\varepsilon & 0 \\ \varepsilon & 1-\varepsilon^2/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\varepsilon^2/2 & -\varepsilon \\ 0 & \varepsilon & 1-\varepsilon^2/2 \end{pmatrix}$$

$$R_y(\phi) = \begin{pmatrix} 1-\varepsilon^2/2 & 0 & \varepsilon \\ 0 & 1 & 0 \\ -\varepsilon & 0 & 1-\varepsilon^2/2 \end{pmatrix}$$

Important: We can show that infinitesimal rotations about different axes commute if we ignore terms of order ε^2 or higher.

Infinitesimal rotation of vectors-b

- We can show that if we keep terms of order ε^2 we get:
- A) For rotation about any axis: $R_{any}(0) = 1$.
- B) $R_x(\varepsilon)R_y(\varepsilon) - R_y(\varepsilon)R_x(\varepsilon) = R_z(\varepsilon^2) - R_{any}(\varepsilon)$

Infinitesimal rotation in quantum mechanics-a

- We need to understand the concept of rotation in quantum mechanics.
- Because rotations affect physical systems, the state ket corresponding to a rotated system is expected to look different from the state ket corresponding to the original unrotated system.
- We assign to a rotation operation R (represented by a 3×3 orthogonal matrix) an operator $D(R)$ in the appropriate ket space such:

$$|\alpha\rangle_R = D(R)|\alpha\rangle$$

- Where $|\alpha\rangle_R$, $|\alpha\rangle$ stand for the kets of the rotated and original system.

Infinitesimal rotation in quantum mechanics-b

- **Be careful:** R acts on a column matrix made up of the three components of a classical vector, while the operator $D(R)$ acts on a state vector in ket space.
- The matrix representation of $D(R)$ depends on the dimensionality N of the particular ket space. For example, for $N=2$, which is appropriate for a spin ½ system with no other degrees of freedom $D(R)$ is represented by a 2×2 matrix, for a spin 1 system by a 3×3 matrix, and so on.

Construction of the operator $D(R)$-a

- Classical mechanics has told us that the angular momentum is the generator of rotations in much the same way as momentum generates translation in space and Hamiltonian generates the time evolution. The appropriate **infinitesimal operators** could be written as:

$$U_E = 1 - iG\varepsilon$$

- Where for an infinitesimal displacement dx' in the x-direction:

$$G \to p_x / \hbar, \qquad \varepsilon \to dx'$$

- While for an infinitesimal time evolution displacement dt:

$$G \to H / \hbar, \qquad \varepsilon \to dt$$

Construction of the operator $D(R)$-b

- In the same spirit the operator for an infinitesimal rotation around the *k*-th axis is given with the help of the *k*-th component of the angular momentum operator as:

$$G \to J_k / \hbar, \qquad \varepsilon \to d\phi$$

- More generally for a rotation about the direction characterized by the unit vector $\hat{\mathbf{n}}$ by an infinitesimal angle $d\phi$ we have:

$$D(\hat{\mathbf{n}}, d\phi) = 1 - i\left(\frac{\mathbf{J}\cdot\hat{\mathbf{n}}}{\hbar}\right)d\phi$$

Since J_k is Hermitian, D is guaranteed to be unitary and reduces to the identity operator as $d\phi \to 0$.

Properties of the operator D(R)

$$Identity: \quad R \cdot 1 = R \Rightarrow D(R) \cdot 1 = D(R)$$

$$Closure: \quad R_1 \cdot R_2 = R_3 \Rightarrow D(R_1) \cdot D(R_2) = D(R_3)$$

$$Inverses: \quad R \cdot R^{-1} = 1 \Rightarrow D(R) \cdot D(R^{-1}) = 1$$

$$R^{-1} \cdot R = 1 \Rightarrow D(R^{-1}) \cdot D(R) = 1$$

$$Associativity: \quad R_1(R_2R_3) = (R_1R_2)R_3 = R_1R_2R_3$$

$$\Rightarrow D(R_1)\left[D(R_2)D(R_3)\right]$$

$$= \left[D(R_1)D(R_2)\right]D(R_3)$$

$$= D(R_1)D(R_2)D(R_3)$$

Fundamental commutation relations

- The whole discussion above can help us to arrive at the fundamental commutation relation for the components of the angular momentum:

$$\left[J_i, J_j\right] = i\hbar\varepsilon_{ijk} J_k$$

From now on in this lecture we use J for either l or s

- We must emphasize that the above relation has been obtained using the following two concepts:
- A) J_k is the generator of rotation about the k-th axis.
- B) Rotation about different axes fail to commute.
- When the generators of infinitesimal transformations do not commute, the corresponding group of operations is said to be **non-Abelian**.

Spin 1/2 systems and finite rotations

- The lowest number N, of dimensions in which the angular-momentum commutation relations are realized is $N=2$. We can prove that the following operator really rotates the system:

$$D_z(\phi) = \exp\left(-\frac{iS_z\phi}{\hbar}\right)$$

- The expectation values of the spin operator behaves as though it were a classical vector under rotation:

$$\langle S_k \rangle \to \sum_i R_{kl} \langle S_l \rangle$$

But the rotation operator has a surprise for a us!

Rotations in Pauli formalism

- With the help of Pauli matrices we write:

$$\exp\left(-\frac{i\mathbf{S}\cdot\hat{\mathbf{n}}\phi}{\hbar}\right)=\exp\left(-\frac{i\hat{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right)$$

which can be written as:

$$\exp\left(-\frac{i\hat{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right)=\mathbf{1}\cos\left(\frac{\phi}{2}\right)-i\hat{\sigma}\cdot\hat{\mathbf{n}}\sin\left(\frac{\phi}{2}\right)$$

- or in a matrix form:

$$\exp\left(-\frac{i\hat{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right)=\begin{pmatrix}\cos\left(\frac{\phi}{2}\right)-in_z\sin\left(\frac{\phi}{2}\right) & \left(-in_x-n_y\right)\sin\left(\frac{\phi}{2}\right)\\ \left(-in_x+n_y\right)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right)+in_z\sin\left(\frac{\phi}{2}\right)\end{pmatrix}$$

The Special Orthogonal Group in 3 dimensions SO(3)

- The set of all multiplication operations with orthogonal matrices forms a group. By this we mean that the following four requirements are satisfied:

1. The product of any two orthogonal matrices is another orthogonal matrix.
2. The associative law holds: $R_1(R_2R_3)=(R_1R_2)R_3$.
3. The identity matrix 1 – physically corresponding to no rotation – defined by $R1=1R=R$ is a member of the class of all orthogonal matrices.
4. The inverse matrix R^{-1} – physically corresponding to rotation in the opposite sense – defined by $RR^{-1}=R^{-1}R=1$ is also a member.