PHYS-505/551

The addition of angular momenta

Lecture-5

Introduction-a

- When we seek to find the total angular momentum in the hydrogen atom we must add the orbital angular momentum **l** with the spin s. For the same purpose in multielectron atoms we need to add the orbital angular momenta and spin of all the electrons.
- The total angular momentum **j** is defined the vector sum orbital angular momentum **l** and the spin **s** as follows:

$$\mathbf{j} = \mathbf{l} + \mathbf{s} \qquad (5.1)$$

Introduction-b

- But now the vectors **l** and **s** are quantum vectors which obey the following permutation relations:

$$[l_z, l_x] = i\hbar l_y, \quad [l_x, l_y] = i\hbar l_z, \quad [l_y, l_z] = i\hbar l_x \qquad (5.2)$$

$$[s_z, s_x] = i\hbar s_y, \quad [s_x, s_y] = i\hbar s_z, \quad [s_y, s_z] = i\hbar s_x \qquad (5.3)$$

- The first step is to show that the vector **j** is also an angular momentum, that means it satisfies similar relations as (4.2) and (4.3).

The total angular momentum-a

- Since the quantity **j**=**l**+**s** is indeed an angular momentum it must obey the following two foundamental relations:

$$\mathbf{j}^2\left|j,m_j\right\rangle = j\left(j+1\right)\hbar^2\left|j,m_j\right\rangle \qquad (5.4)$$

$$j_z\left|j,m_j\right\rangle = m_j\hbar\left|j,m_j\right\rangle \qquad (5.5)$$

- Where m_j for a given j will get $2j+1$ values

$$m_j = -j, \;\ldots\; , +j \qquad (5.6)$$

The total angular momentum-b

- Now the question comes naturally: *What are the possible values of j for given l and s?*
- The answer is: *For given l and s the angular momentum j takes the values:*

$$j = |l - s|, \underbrace{\; . \; . \; . \; . \; . \;}_{\text{unit step}}, l + s$$

The proof is given in the class

The eigenstates of the total angular momentum-a

- With the following examples we show the idea of constructing the total angular momentum eigenstates:
- A) Construct the states of definite total angular momentum of a hydrogen atom at the state 2*p*.
- B) Construct the state of definite total spin for two particles with spin 1/2 each.

Examples will be solved in class

The common eigenstates of J_1^2, J_2^2, J_{1z}, J_{2z}

- In general let's consider two different angular momenta $\mathbf{j}_1$, $\mathbf{j}_2$. These momenta can be angular momenta relating two different particles or angular momenta relating to one particle (for example, orbital angular momentum and spin).
- These two momenta act in different state spaces, so that all their components are commuting with one another. The individual states of $\mathbf{j}_1$, $\mathbf{j}_2$ will be denoted, as usual,

$$\left| j_1 m_1 \right\rangle, \; \left| j_2 m_2 \right\rangle$$

The common eigenstates of J_1^2, J_2^2, J_{1z}, J_{2z}

- For this states we have the usual properties:

$$\begin{cases} \mathbf{j}_1^2 \left| j_1 m_1 \right\rangle = \hbar^2 j_1 \left(j_1 + 1 \right) \left| j_1 m_1 \right\rangle \\ j_{1z} \left| j_1 m_1 \right\rangle = \hbar m \left| j_1 m_1 \right\rangle \end{cases} \qquad (5.7)$$

(similarly for the particle 2)

- The operators $\mathbf{j}_1^2, j_{1z}$ can be represented in the base $\left| j_1 m_1 \right\rangle$ with square matrices of dimensions (the same for the particle 2).

The common eigenstates of J_1^2, J_2^2, J_{1z}, J_{2z}

- When the two particles make up a system **we must be careful**. The state space of the compound system is obtained by taking the direct product (or tensor product) of the individual state space of the two angular momenta. The eigenvectors of the new space are denoted as:

$$\left|j_1 m_1\right\rangle \otimes \left|j_2 m_2\right\rangle = \left|j_1 m_1\right\rangle \left|j_2 m_2\right\rangle = \left|j_1 j_2 \; ; \; m_1 m_2\right\rangle \equiv \left|m_1 m_2\right\rangle \tag{5.8}$$

- These eigenstates are orthonormal and make up a full base.

The common eigenstates of $J_1^2, J_2^2, J_{1z}, J_{2z}$

- For fixed j_1, j_2, m_1 and m_2 have the values (integer or half-integers):

$$\begin{cases} m_1 = -j_1, -j_1+1, \ \ldots j_1 \\ m_2 = -j_2, -j_2+1, \ \ldots j_2 \end{cases} \qquad (5.9)$$

- The state space of the compound system is a $(2j_1+1)(2j_2+1)$ -dimensional space.
- The states $\left| m_1 m_2 \right\rangle$ are, according to their construction, eigenstates of the operators

$$\left\{ \mathbf{j}_1^2, \ \mathbf{j}_2^2, \ j_{1z}, \ j_{2z} \right\}$$

The common eigenstates of J_1^2, J_2^2, J_{1z}, J_{2z}

- These states satisfy the following properties:

$$\mathbf{j}_1^2 \left| j_1 j_2 m_1 m_2 \right\rangle = \hbar^2 j_1 \left(j_1 + 1 \right) \left| j_1 j_2 m_1 m_2 \right\rangle \quad (5.10a)$$

$$j_{1z} \left| j_1 j_2 m_1 m_2 \right\rangle = \hbar m_1 \left| j_1 j_2 m_1 m_2 \right\rangle \quad (5.10b)$$

$$\mathbf{j}_2^2 \left| j_1 j_2 m_1 m_2 \right\rangle = \hbar^2 j_2 \left(j_2 + 1 \right) \left| j_1 j_2 m_1 m_2 \right\rangle \quad (5.10c)$$

$$j_{2z} \left| j_1 j_2 m_1 m_2 \right\rangle = \hbar m_2 \left| j_1 j_2 m_1 m_2 \right\rangle \quad (5.10d)$$

The eigenstates of the total angular momentum-The common eigenstates of J_1^2, J_2^2, J^2, J_z

- In the absence of interaction between $\mathbf{j}_1$, $\mathbf{j}_2$, the operators $\mathbf{j}_1$, $\mathbf{j}_2$ commute with the total Hamiltonian and thus $|j_1 m_1\rangle, |j_2 m_2\rangle$ are also eigenstates of the system. But what happens if the there is an interaction between $\mathbf{j}_1$, $\mathbf{j}_2$?
- In this case $\mathbf{j}_1$, $\mathbf{j}_2$ are not conserved but $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$ is conserved. It is better then to transform to an eigenstate basis of the operators

$$\left\{\mathbf{j}_1^2,\ \mathbf{j}_2^2,\ \mathbf{J}^2,\ J_z\right\}$$

The eigenstates of the total angular momentum-The common eigenstates of J_1^2, J_2^2, J^2, J_z

- The eigenstates in this basis will be denoted by $\left|j_1\, j_2\; J\; M\right\rangle \equiv \left|J\; M\right\rangle$ and satisfy

$$\begin{cases} \mathbf{J}^2\left|JM;j_1j_2\right\rangle = \hbar^2 J\left(J+1\right)\left|JM;j_1j_2\right\rangle \\ J_z\left|JM;j_1j_2\right\rangle = \hbar M\left|JM;j_1j_2\right\rangle \\ J_1^2\left|JM;j_1j_2\right\rangle = \hbar^2 j_1\left(j_1+1\right)\left|JM;j_1j_2\right\rangle \\ J_2^2\left|JM;j_1j_2\right\rangle = \hbar^2 j_2\left(j_2+1\right)\left|JM;j_1j_2\right\rangle \end{cases} \quad (5.11)$$

$$J = \left|j_1 - j_2\right|,\ \left|j_1 - j_2\right| + 1, \underbrace{\ldots\ldots}_{\text{unit step}}, j_1 + j_2 \quad (5.12)$$

$$M = -J,\ -J+1,\ \ldots,\ J$$

The eigenstates of the total angular momentum-The common eigenstates of J_1^2, J_2^2, J^2, J_z

- The eigenstates $\left| j_1 j_2\, J\, M \right\rangle \equiv \left| J\, M \right\rangle$ satisfy also

$$J = \left| j_1 - j_2 \right|,\ \left| j_1 - j_2 \right| + 1, \underbrace{\ldots\ldots}_{\text{unit step}}, j_1 + j_2 \quad (5.13)$$

$$M = -J,\ -J+1,\ \ldots,\ J$$

- And of course they form a complete basis since

$$\left\langle J' M' \middle| JM \right\rangle = \delta_{J'J}\,\delta_{M'M} \qquad \sum_{J} \sum_{M=-J}^{J} \left| JM \right\rangle \left\langle JM \right| = 1 \quad (5.14)$$

The states $\left| J\, M \right\rangle$ do not have specific values of m_1, m_2. But it holds that $M=m_1+m_2$

The Clebsch-Gordan coefficients-c

- The two sets of orthonormal states $|m_1 m_2\rangle$ and $|JM\rangle$ are related by a unitary transform; that is we can write $|JM\rangle$ in terms of $|m_1 m_2\rangle$ as follows

$$|JM\rangle = \sum_{m_1, m_2} \langle m_1 m_2 \,|\, JM\rangle |m_1 m_2\rangle \qquad (5.15)$$

- The terms $c_{m_1 m_2} = \langle m_1 m_2 \,|\, JM\rangle$ are known as *Clebsch-Gordan coefficients*. They are simply the elements of the transformation matrix that connects the $|m_1 m_2\rangle$ to the $|JM\rangle$ basis
- This means that, from the linear combination to get a state with not only a definite *M* but also with a definite *J*.

The Clebsch-Gordan coefficients-d

- It is possible to obtain a general expression for the C-G coefficients. However it is simpler to construct the coefficients for particular cases. They can be calculated by successive applications of $J_{\pm} = J_x \pm iJ_y$ on the vectors $|JM\rangle$ as follows:

$$\begin{cases} J_{\pm}|JM\rangle = \hbar\sqrt{J(J+1) - M(M+1)}\,|J, M \pm 1\rangle \\ J_{1\pm}|m_1 m_2\rangle = \hbar\sqrt{J_1(J_1+1) - m_1(m_1+1)}\,|m_1 \pm 1, m_2\rangle \end{cases} \quad (5.16)$$

- Together with the relation:

$$\left|J = J_1 + J_2, M = \pm\left(j_1 + j_2\right)\right\rangle = \left|m_1 = \pm j_1,\ m_2 = \pm j_1\right\rangle \quad (5.17)$$

Properties of the Clebsch-Gordan coefficients-a

- $$\langle m_1, m_2 | JM \rangle = 0 \qquad \text{unless} \quad M = m_1 + m_2 \qquad (5.18)$$

  $$\langle m_1, m_2 | JM \rangle = \text{ is real} \qquad (5.19)$$

  $$\sum_{m_1=-j_1}^{m_1=j_1} \sum_{m_2=-j_2}^{m_2=j_2} \langle JM | m_1, m_2 \rangle \langle m_1, m_2 | J' M' \rangle = \delta_{JJ'} \delta_{MM'} \qquad (5.20)$$

  $$\sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-j}^{J} \langle m_1, m_2 | JM \rangle \langle JM | m_1', m_2' \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \qquad (5.21)$$

Properties of the Clebsch-Gordan coefficients-b

$$\sqrt{J(J+1)-M(M+1)}\left\langle m_1 m_2 \middle| J, M+1\right\rangle = \tag{5.22}$$

$$\sqrt{j_1(j_1+1)-m_1(m_1+1)}\left\langle m_1 \mp 1, m_2 \middle| JM\right\rangle + \sqrt{j_2(j_2+1)-m_2(m_2+1)}\left\langle m_1, m_2-1 \middle| JM\right\rangle$$

$$\left\langle m_2 m_1 \middle| JM\right\rangle = \left(-1\right)^{j_1+j_2-J}\left\langle m_1 m_2 \middle| JM\right\rangle \tag{5.23}$$

$$\left\langle -m_1, -m_2 \middle| J, -M\right\rangle = \left(-1\right)^{j_1+j_2-J}\left\langle m_1 m_2 \middle| JM\right\rangle \tag{5.24}$$