Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS
MATH 1110

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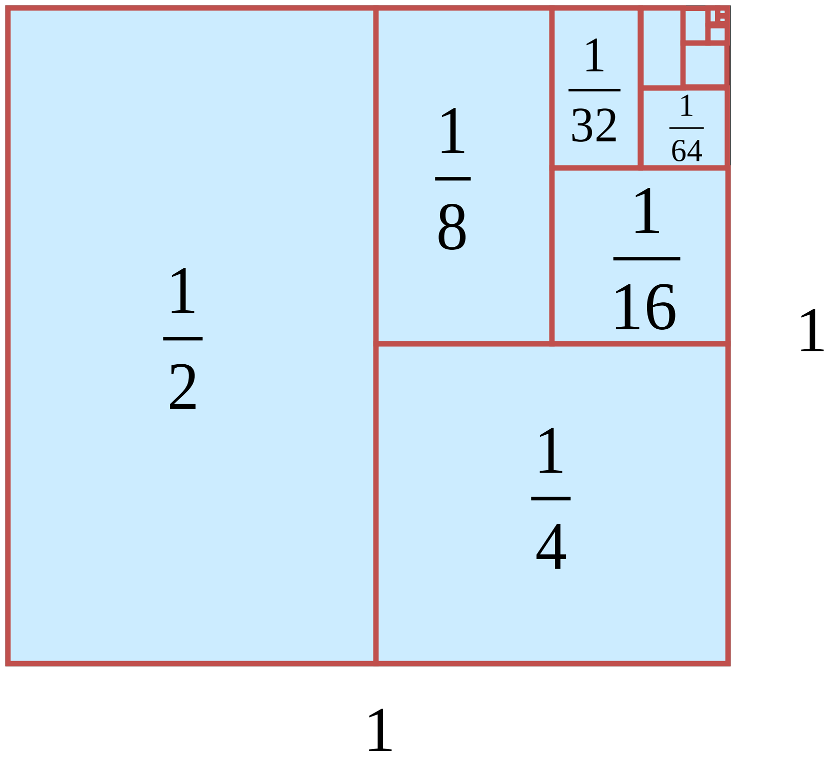
Start with a square one unit by one unit:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots=1$$



This is an example of an infinite series.

This series converges (approaches a limiting value.)

Many series do not converge: $$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots=\infty$$

In an infinite series: $a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$

$a_1, a_2, \ldots$ are terms of the series. a_n is the n^{th} term.

Partial sums:

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_n = \sum_{k=1}^{n} a_k$$

n^{th} partial sum

If S_n has a limit as $n \to \infty$, then the series converges, otherwise it diverges.

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Geometric Series:

In a geometric series, each term is found by multiplying the preceding term by the same number, r.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

This converges to $\dfrac{a}{1-r}$ if $|r| < 1$, and diverges if $|r| \geq 1$.

$-1 < r < 1$ is the interval of convergence.

$\rightarrow$

Example 1:

$$\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\cdots$$

$$.3+.03+.003+.0003+\cdots=.333\ldots$$

$$=\frac{1}{3}$$

$$\frac{\overbrace{\frac{3}{10}}^{a}}{1-\underbrace{\frac{1}{10}}_{r}}=\frac{\frac{3}{10}}{\frac{9}{10}}=\frac{3}{9}=\frac{1}{3}$$

$\rightarrow$

Example 2:

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$$

$$\frac{\overset{a}{1}}{1-\underset{r}{\left(-\frac{1}{2}\right)}}=\frac{1}{1+\frac{1}{2}}=\frac{1}{\frac{3}{2}}=\frac{2}{3}$$

→

The partial sum of a geometric series is: $S_n = \dfrac{a\left(1-r^n\right)}{1-r}$

If $\left|r\right|<1$ then $\displaystyle\lim_{n\to\infty}\frac{a\left(1-r^{n}\right)}{1-r}=\frac{a}{1-r}$

If $\left|x\right|<1$ and we let $r=x$, then:

$$1+x+x^2+x^3+\cdots=\frac{1}{1-x}$$

The more terms we use, the better our approximation (over the interval of convergence.)

$\rightarrow$

Power Series

Power Series

- In the previous sections we concentrated on the infinite series with constant terms.

- The most importance in applications are series whose series terms contain variables, as in the next definition.

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Definition (11.33)

Let x be a variable. A **power series in x** is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

where each a_i is a real number.

If a number is substituted for x in Definition (11.33) we obtain a series of constant terms which may be tested for convergence or divergence. To simplify the general term of the power series it is assumed that $x^0 = 1$ even if $x = 0$. The main objective of this section is to determine all values of x for which a power series converges. Evidently, every power series in x converges if $x = 0$. To find other numbers that produce a convergent series we often employ the Ratio Test, as illustrated in the following examples.

Example 1 Find all values of x for which the following power series is absolutely convergent:

$$1 + \frac{1}{5}x + \frac{2}{5^2}x^2 + \cdots + \frac{n}{5^n}x^n + \cdots$$

Solution If we let $u_n = (n/5^n)x^n = nx^n/5^n$, then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{5^{n+1}}\cdot\frac{5^n}{nx^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)x}{5n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{5n}\right)|x| = \frac{1}{5}|x|.\end{aligned}$$

By the Ratio Test, the series is absolutely convergent if the following equivalent inequalities are true:

$$\frac{1}{5}|x| < 1, \quad |x| < 5, \quad -5 < x < 5.$$

The series diverges if $\frac{1}{5}|x| > 1$, that is, if $x > 5$ or $x < -5$. If $\frac{1}{5}|x| = 1$, the Ratio Test gives no information and hence the numbers 5 and -5 require special consideration. Substituting 5 for x in the power series we obtain

$$1 + 1 + 2 + 3 + \cdots + n + \cdots$$

which is divergent. If we let $x = -5$ we obtain

$$1 - 1 + 2 - 3 + \cdots + n(-1)^n + \cdots$$

which is also divergent. Consequently, the power series is absolutely convergent for every x in the open interval $(-5, 5)$ and diverges elsewhere. ■

Example 2 Find all values of x for which the following power series is absolutely convergent:

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$

Solution We shall employ the same technique used in Example 1. If we let

$$u_n = \frac{1}{n!} x^n = \frac{x^n}{n!}$$

then

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = \lim_{n\to\infty} \frac{1}{n+1} |x| = 0.$$

Since the limit 0 is less than 1 for every value of x, it follows from the Ratio Test that the power series is absolutely convergent for all real numbers. ■

Example 3 Find all values of x for which $\sum n!x^n$ is convergent.

Solution Let $u_n = n!x^n$. If $x \neq 0$, then

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right|$$
$$= \lim_{n \to \infty} |(n+1)x| = \lim_{n \to \infty} (n+1)|x| = \infty$$

and, by the Ratio Test, the series diverges. Hence the power series is convergent only if $x = 0$. ■