Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS
MATH 1110

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Theorem (11.34)

(i) If a power series $\sum a_n x^n$ converges for a nonzero number c, then it is absolutely convergent whenever $|x| < |c|$.

(ii) If a power series $\sum a_n x^n$ diverges for a nonzero number d, then it diverges whenever $|x| > |d|$.

By applying theorem 11.14

Theorem (11.14)

If an infinite series $\sum a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$.

Definition (11.3)

A sequence $\{a_n\}$ **has the limit** L, written

$$\lim_{n \to \infty} a_n = L$$

if for every $\varepsilon > 0$, there exists a positive number N such that

$$|a_n - L| < \varepsilon \quad \text{whenever } n > N.$$

Proof If $\sum a_n c^n$ converges and $c \neq 0$, then by Theorem (11.14), $\lim_{n\to\infty} a_n c^n = 0$. Employing Definition (11.3) with $\varepsilon = 1$, there is a positive integer N such that

$$|a_n c^n| < 1 \quad \text{whenever } n \geq N$$

and, therefore,

$$|a_n x^n| = \left|\frac{a_n c^n x^n}{c^n}\right| = |a_n c^n| \left|\frac{x}{c}\right|^n < \left|\frac{x}{c}\right|^n$$

provided $n \geq N$, If $|x| < |c|$, then $|x/c| < 1$ and $\sum |x/c|^n$ is a convergent geometric series. Hence, by the Comparison Test (11.24),

Basic Comparison Test (11.24)

Suppose $\sum a_n$ and $\sum b_n$ are positive term series.

(i) If $\sum b_n$ converges and $a_n \leq b_n$ for every positive integer n, then $\sum a_n$ converges.

(ii) If $\sum b_n$ diverges and $a_n \geq b_n$ for every positive integer n, then $\sum a_n$ diverges.

provided $n \geq N$, If $|x| < |c|$, then $|x/c| < 1$ and $\sum |x/c|^n$ is a convergent geometric series. Hence, by the Comparison Test (11.24), the series obtained by deleting the first N terms of $\sum |a_n x^n|$ is convergent. It follows that the series $\sum |a_n x^n|$ is also convergent, which proves (i).

To prove (ii), suppose the series diverges for $x = d \neq 0$. If the series converges for some c_1, where $|c_1| > |d|$, then by (i) it converges whenever $|x| < |c_1|$. In particular it converges for $x = d$, contrary to our supposition. Hence the series diverges whenever $|x| > |d|$. □

Theorem (11.35)

> If $\sum a_n x^n$ is a power series, then precisely one of the following is true.
>
> (i) The series converges only if $x = 0$.
>
> (ii) The series is absolutely convergent for all x.
>
> (iii) There is a positive number r such that the series is absolutely convergent if $|x| < r$ and divergent if $|x| > r$.

Proof If neither (i) nor (ii) is true, then there exist nonzero numbers c and d such that the series converges if $x = c$ and diverges if $x = d$. Let S denote the set of all real numbers for which the series is absolutely convergent. By Theorem (11.34), the series diverges if $|x| > |d|$ and hence every number in S is less than $|d|$. By the Completeness Property (11.9), S has a least upper bound r. It follows that the series is absolutely convergent if $|x| < r$ and diverges if $|x| > r$. □

A sequence is said to be **monotonic** if successive terms are nondecreasing in the sense that

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$$

or nonincreasing in the sense that

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots$$

A sequence is **bounded** if there is a positive real number M such that $|a_k| \leq M$ for all k. The next theorem is fundamental for later developments.

Theorem (11.8) A bounded, monotonic, infinite sequence has a limit.

To prove Theorem (11.8), it is necessary to use an important property of real numbers. Let us first state several definitions. If S is a nonempty set of real numbers, then a real number u is called an **upper bound** of S if $x \leq u$ for

every x in S. A number v is a **least upper bound** of S if v is an upper bound and no number less than v is an upper bound of S. The least upper bound is, therefore, the smallest real number that is greater than or equal to every number in S. To illustrate, if S is the open interval (a, b), then any number greater than b is an upper bound of S; however, the least upper bound of S is unique, and equals b (see Exercise 50).

The following statement is an axiom for the real number system.

The Completeness Property (11.9)

If a nonempty set S of real numbers has an upper bound, then S has a least upper bound.

If (iii) of Theorem (11.35) occurs, then the power series $\sum a_n x^n$ is absolutely convergent throughout the open interval $(-r, r)$ and diverges outside of the closed interval $[-r, r]$, as illustrated in Figure 11.4. The number r is called the **radius of convergence** of the series. Either convergence or divergence may occur at $-r$ or r, depending on the nature of the series.

The totality of numbers for which a power series converges is called its **interval of convergence**. If the radius of convergence r is positive, then the interval of convergence is one of the following:

$$(-r, r), \quad (-r, r], \quad [-r, r), \quad \text{or} \quad [-r, r].$$

If (i) or (ii) of Theorem (11.35) occurs, then the radius of convergence is denoted by 0 or ∞, respectively. In Example 1 of this section, the interval of convergence is $(-5, 5)$ and the radius of convergence is 5.

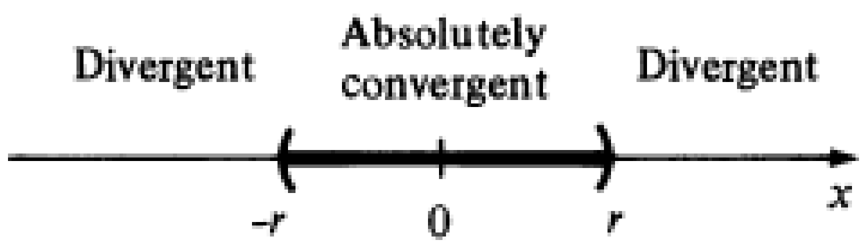

FIGURE 11.4 $\sum a_n x^n$, radius of convergence r.

Example Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x^n$.

Solution Note that in this example the coefficient of x^0 is 0 and the summation begins with $n = 1$. We let $u_n = x^n/\sqrt{n}$ and consider

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right| = \lim_{n\to\infty} \left| \frac{\sqrt{n}}{\sqrt{n+1}} x \right|$$

$$= \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} \, |x| = (1)|x| = |x|.$$

It follows from the Ratio Test that the power series is absolutely convergent if $|x| < 1$, that is, if $-1 < x < 1$. The series diverges if $|x| > 1$, that is, if $x > 1$

or $x < -1$. The numbers 1 and -1 must be checked by direct substitution in the power series. If we let $x = 1$ we obtain

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(1)^n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$$

which is a divergent p-series. If we substitute $x = -1$ the result is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(-1)^n = -1 + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{3}} + \cdots + \frac{(-1)^n}{\sqrt{n}} + \cdots$$

which converges by the Alternating Series Test. Hence the power series converges if $-1 \leq x < 1$, and the interval of convergence is $[-1, 1)$. ■

Definition (11.36)

> Let c be a real number and x a variable. A **power series in $x - c$** is a series of the form
>
> $$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$
>
> where each a_i is a real number.

To simplify the nth term in (11.36), it is assumed that $(x - c)^0 = 1$ even if $x = c$. If we employ the same reasoning used to prove Theorem (11.35), and replace x by $x - c$, then it can be shown that precisely one of the following is true.

(i) The series converges only if $x - c = 0$, that is, if $x = c$.

(ii) The series is absolutely convergent for all x.

(iii) There is a positive number r such that the series is absolutely convergent if $|x - c| < r$ and divergent if $|x - c| > r$.

If (iii) occurs, then the series $\sum a_n(x-c)^n$ is absolutely convergent if

$$-r < x - c < r, \quad \text{or} \quad c - r < x < c + r$$

that is, if x is the interval $(c - r, c + r)$ as illustrated in Figure 11.5. The endpoints of the interval must be checked separately. As before, the totality of numbers for which the series converges is called the **interval of convergence**, and r is called the **radius of convergence**.

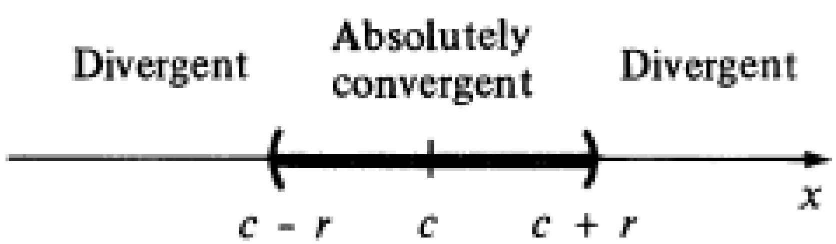

FIGURE 11.5 $\sum a_n(x-c)^n$, radius of convergence r.

Example Find the interval of convergence of

$$1 - \frac{1}{2}(x-3) + \frac{1}{3}(x-3)^2 + \cdots + (-1)^n \frac{1}{n+1}(x-3)^n + \cdots$$

Solution If we let $u_n = (-1)^n(x-3)^n/(n+1)$, then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+2}\cdot\frac{n+1}{(x-3)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{n+1}{n+2}(x-3)\right| \\ &= \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)|x-3| \\ &= (1)|x-3| = |x-3|.\end{aligned}$$

By the Ratio Test the series is absolutely convergent if $|x-3| < 1$, that is, if

$$-1 < x - 3 < 1 \quad \text{or} \quad 2 < x < 4.$$

The series diverges if $x < 2$ or $x > 4$. The numbers 2 and 4 must be checked separately. If $x = 4$ the resulting series is

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^n \frac{1}{n+1} + \cdots$$

which converges by the Alternating Series Test. For $x = 2$ the series becomes

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} + \cdots$$

which is the divergent harmonic series. Hence the interval of convergence is $(2, 4]$. ■

Alternating Series

An infinite series whose terms are alternately positive and negative is called an **alternating series**. It is customary to express an alternating series in one of the forms

$$a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1}a_n + \cdots$$

or

$$-a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n + \cdots$$

Example :

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

SOLUTION If $a_n = n(x+2)^n/3^{n+1}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right|$$

$$= \left(1 + \frac{1}{n}\right)\frac{|x+2|}{3} \rightarrow \frac{|x+2|}{3} \qquad \text{as } n \rightarrow \infty$$

Using the Ratio Test, we see that the series converges if $|x + 2|/3 < 1$ and it diverges if $|x + 2|/3 > 1$. So it converges if $|x + 2| < 3$ and diverges if $|x + 2| > 3$. Thus the radius of convergence is $R = 3$.

The inequality $|x + 2| < 3$ can be written as $-5 < x < 1$, so we test the series at the endpoints -5 and 1. When $x = -5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence [$(-1)^n n$ doesn't converge to 0]. When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus the series converges only when $-5 < x < 1$, so the interval of convergence is $(-5, 1)$.