PHYS-454

Laguerre Polynomials

Lecture-7

Laguerre Polynomials

- The **Laguerre polynomials**, named after Edmond Laguerre (1834 - 1886), are solutions of **Laguerre's equation:**

$$xy'' + (1 - x)y' + ny = 0$$

- which is a second-order linear differential equation. This equation has nonsingular solutions only if n is a non-negative integer.

Laguerre Polynomials

- The Laguerre polynomials are generated from the following Rodriguez formula:

$$L_n(x) = \frac{e^x}{n!}\left(\frac{d}{dx}\right)^n \left(e^{-x}x^n\right)$$

$$L_0 = 1$$

$$L_1 = -x + 1$$

$$L_2 = \frac{1}{2}\left(x^2 - 4x + 2\right)$$

$$L_3 = \frac{1}{6}\left(-x^3 + 9x^2 - 18x + 6\right)$$

Laguerre Polynomials

- The Laguerre polynomials can be given in a series form as follows:

$$L_n\left(x\right) = \sum_{k=0}^{n} \frac{\left(-1\right)^{n-k} n!}{k!(n-k)!(n-k)!} x^{n-k}$$

- They are also produced by the following generating function

$$\frac{1}{1-t} e^{-\frac{tx}{1-t}} = \sum_{k=0}^{n} L_n(x) t^n$$

Laguerre Polynomials

- The Laguerre polynomials satisfy the following recurrence relations:

$$L_{n+1}(x) = \frac{(2n+1-x)L_n(x) - nL_{n-1}(x)}{n+1}$$

$$xL_n^{'}(x) = nL_n(x) - nL_{n-1}(x)$$

$$L_{n-1}(x) = L_{n-1}^{'}(x) - L_n^{'}(x)$$

$$L_n(0) = 1$$

Laguerre Polynomials

- The Laguerre polynomials do not by themselves form an orthogonal set. However they satisfy the following property:

$$\int_0^\infty e^{-x} L_m(x) L_n(x)\, dx = \delta_{m,n}$$

- By defining $\varphi_n(x) = e^{-x/2} L_n(x)$ we get an orthonormal function which satisfies the self-adjoint diff. equation

$$x\varphi_n'' + \varphi_n' + \left(n + \frac{1}{2} - \frac{x}{4}\right)\varphi_n = 0$$

Associated Laguerre Polynomials

- The associated Laguerre y polynomials are defined by:

$$L_n^k(x) \equiv (-1)^k \left(\frac{d}{dx}\right)^k L_{n+k}(x)$$

$$L_0^k(x) = 1$$

$$L_1^k = -x + k + 1$$

$$L_2^k = \frac{x^2}{2} - (k+2)x + \frac{(k+2)(k+1)}{2}$$

Associated Laguerre Polynomials

- The associated Laguerre polynomials are given in a series form as follows:

$$L_n^k\left(x\right)=\sum_{m=0}^{n}\frac{\left(-1\right)^m\left(n+k\right)!}{m!(n-m)!(k+m)!}x^m,\quad k>-1$$

- The associated Laguerre polynomials are produced from the following generating function:

$$\frac{1}{\left(1-t\right)^{k+1}}e^{-\frac{tx}{1-t}}=\sum_{n=0}^{\infty}L_n^k(x)t^n$$

Associated Laguerre Polynomials

- The associated Laguerre polynomials satisfy the following recurrence relations:

$$(n+1)L_{n+1}^{k}(x) = (2n+k+1-x)L_{n}^{k}(x) - (n+k)L_{n-1}^{k}(x)$$

$$xL_{n}^{k'}(x) = nL_{n}^{k}(x) - (n+k)L_{n-1}^{k}(x)$$

$$L_{n}^{k}(0) = \frac{(n+k)!}{n!k!}$$

Associated Laguerre Polynomials

- The associated Laguerre polynomials satisfy the following differential equation:

$$xL_n^{k''}(x) + (k+1-x)L_n^{k'}(x) + nL_n^k(x) = 0$$

- We also may define a Rodrigues representation:

$$L_n^k(x) = \frac{e^x x^{-k}}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+k}\right)$$

Associated Laguerre Polynomials

- The associated Laguerre equations is not self-adjoint. It can be put in such form by multiplying by $e^{-x}x^{k}$:

$$\int_0^\infty e^{-x}x^k L_n^k(x)L_m^k(x)dx = \frac{(n+k)!}{n!}\delta_{m,n}$$

$$\psi_n^k(x) = e^{-x/2}x^{k/2}L_n^k(x)$$

$$x\psi_n^{k\,''} + \psi_n^{k\,'} + \left(\frac{2n+k+1}{2} - \frac{k^2}{4x} - \frac{x}{4}\right)\psi_n^k = 0$$

Associated Laguerre Polynomials

- A further useful form can be obtained by defining:

$$\Phi_n^k(x) = e^{-x/2} x^{(k+1)/2} L_n^k(x)$$

$$\Phi_n^{k\,''} + \left(\frac{2n+k+1}{2} - \frac{k^2-1}{4x^2} - \frac{1}{4} \right) \Phi_n^k = 0$$

$$\int_0^\infty e^{-x} x^{k+1} L_n^k(x) L_m^k(x)\,dx = \frac{(n+k)!}{n!}(2n+k+1)$$