Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS
MATH 1110

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Taylor and Maclaurin Series

Suppose a function f is represented by a power series in $x - c$, such that

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$
$$= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \cdots$$

By differentiating

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$
$$= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-c)^{n-2}$$
$$= 2a_2 + (3\cdot 2)a_3(x-c) + (4\cdot 3)a_4(x-c)^2 + \cdots$$

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)a_n(x-c)^{n-3}$$
$$= (3\cdot 2)a_3 + (4\cdot 3\cdot 2)a_4(x-c) + \cdots$$

$$
\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \\
&= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n (x-c)^{n-2} \\
&= 2a_2 + (3\cdot 2)a_3(x-c) + (4\cdot 3)a_4(x-c)^2 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
f'''(x) &= \sum_{n=3}^{\infty} n(n-1)(n-2) a_n (x-c)^{n-3} \\
&= (3\cdot 2)a_3 + (4\cdot 3\cdot 2)a_4(x-c) + \cdots
\end{aligned}
$$

and, for every positive integer k,

$$
f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x-c)^{n-k}.
$$

Substituting c for x in each of these series representations, we obtain :

$$f(c) = a_0, \quad f'(c) = a_1, \quad f''(c) = 2a_2, \quad f'''(c) = (3 \cdot 2)a_3$$

and, for every positive integer n,

$$f^{(n)}(c) = n!a_n, \quad \text{or} \quad a_n = \frac{f^{(n)}(c)}{n!}.$$

We have proved the following result.

Theorem (11.38)

If f is a function and

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

for all x in an open interval containing c, then

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

The series that appears in the conclusion of Theorem (11.38) is called the **Taylor series for $f(x)$ at c**. The special case $c = 0$, stated in the following corollary, is extremely important.

Corollary (11.39)

If f is a function and $f(x) = \sum a_n x^n$ for all x in an open interval $(-r, r)$, then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

The series in this corollary is called the **Maclaurin series for $f(x)$**. Each example in the preceding section involves a Maclaurin series.

EXAMPLE 1

Find the Maclaurin series for $\sin x$

Corollary (11.39)

If f is a function and $f(x) = \sum a_n x^n$ for all x in an open interval $(-r, r)$, then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

The series in this corollary is called the **Maclaurin series for $f(x)$**. Each example in the preceding section involves a Maclaurin series.

Solution Let us arrange our work as follows:

$$\begin{array}{ll} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f'''(x) = -\cos x & f'''(0) = -1 \end{array}$$

Successive derivatives follow this same pattern. Substitution in Corollary (11.39) yields the Maclaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

Corollary (11.39)

> If f is a function and $f(x) = \sum a_n x^n$ for all x in an open interval $(-r, r)$, then
>
> $$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

The series in this corollary is called the **Maclaurin series for $f(x)$**. Each example in the preceding section involves a Maclaurin series.

Example 2:

Find a Maclaurin series which represents e^x

Corollary (11.39)

If f is a function and $f(x) = \sum a_n x^n$ for all x in an open interval $(-r, r)$, then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

The series in this corollary is called the **Maclaurin series for $f(x)$**. Each example in the preceding section involves a Maclaurin series.

Solution If $f(x) = e^x$, then $f^{(n)}(x) = e^x$ for every positive integer n. Hence $f^{(n)}(0) = 1$ and substitution in Corollary (11.39) gives us

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Corollary (11.39)

If f is a function and $f(x) = \sum a_n x^n$ for all x in an open interval $(-r, r)$, then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

The series in this corollary is called the **Maclaurin series for $f(x)$**. Each example in the preceding section involves a Maclaurin series.

Example 3:

Find the Taylor series for $\sin x$ in powers of $x - \pi/6$.

We have proved the following result.

Theorem (11.38) If f is a function and

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

for all x in an open interval containing c, then

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

The series that appears in the conclusion of Theorem (11.38) is called the **Taylor series for $f(x)$ at c**. The special case $c = 0$, stated in the following corollary, is extremely important.

Solution 3:

$$f(x) = \sin x \qquad f\left(\frac{\pi}{6}\right) = \frac{1}{2},$$

$$f'(x) = \cos x \qquad f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},$$

$$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{6}\right) = -\frac{1}{2},$$

$$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2},$$

for all x in an open interval containing c, then

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

and this pattern of four numbers repeats itself indefinitely. Substitution in Theorem (11.38) gives us

$$\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2(2!)}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2(3!)}\left(x - \frac{\pi}{6}\right)^3 + \cdots$$

EXAMPLE 5

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution We need to find $f(2), f'(2), f''(2), \ldots$. Taking derivatives we get

$$
\begin{aligned}
f(x) &= x^{-1}, & f(2) &= 2^{-1} = \frac{1}{2}, \\
f'(x) &= -x^{-2}, & f'(2) &= -\frac{1}{2^2}, \\
f''(x) &= 2!x^{-3}, & \frac{f''(2)}{2!} &= 2^{-3} = \frac{1}{2^3}, \\
f'''(x) &= -3!x^{-4}, & \frac{f'''(2)}{3!} &= -\frac{1}{2^4}, \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-1)^n n! x^{-(n+1)}, & \frac{f^{(n)}(2)}{n!} &= \frac{(-1)^n}{2^{n+1}}.
\end{aligned}
$$

The Taylor series is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots.$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ and its sum is

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

In this example the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x-2| < 2$ or $0 < x < 4$. ■

$S_\infty = a_1 / (1-r)$

We have proved the following result.

Theorem (11.38) If f is a function and

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

for all x in an open interval containing c, then

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

The series that appears in the conclusion of Theorem (11.38) is called the **Taylor series for $f(x)$ at c**. The special case $c = 0$, stated in the following corollary, is extremely important.