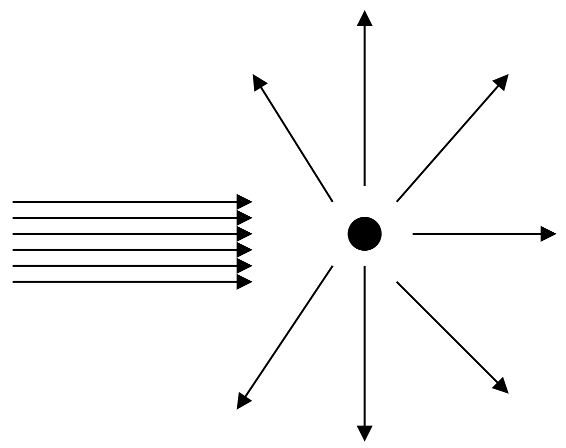
PHYS-505/551

Scattering Theory

Lecture-9

Introduction-a

- Scattering is a problem of motion in the free space. In this case the energy spectrum is continuous. A naive graphical representation of this problem is given by the following picture.



Introduction-b

- The proper tools for describing such a problem are the wavefunctions of the continuous spectrum (which as you remember have a form of plane wave exp(ikz)).
- The key point here is the choice of a suitable *boundary* (or *asymptotic*) *condition* which will describe the picture of the previous transparency: *the scattered wave will be made up by waves that are outgoing with respect to the scattering center.*

Introduction-c

- For our study we are going to make also the following assumptions:

1. Any interaction between the scattered particles themselves is neglected.
2. Possible multiple scattering processes are neglected. A multiple scattering process is a process in which a scattered particle can be scattered multiple times in the same target range.
3. The incident beam width is much larger than a typical range of the scattering potential, so that the particles will have a well-defined momentum.

Introduction-d

- The mathematical form of these functions at very large distances away from the scattering center will assume the form:

$$\psi \rightarrow e^{ikz} + f\left(\theta\right)\frac{e^{ikr}}{r} \qquad (9.1)$$

- The above relation expresses mathematically the following physical picture: The outgoing wave is made up by two different terms, the first corresponds to the *incident wave,* the second corresponds to the *scattered wave*.

Introduction-e

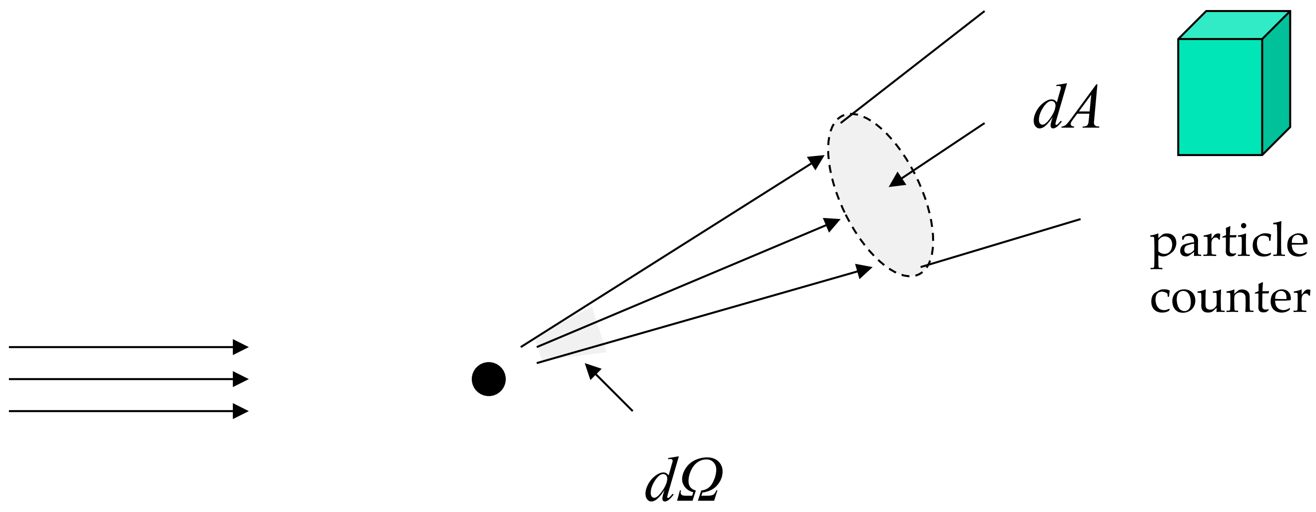
- Why the scattered wave has such a form?
- Because the scattered wave propagates isotropically in all the directions of space. Thus, its amplitude must tend to zero as the distance from the scattering center gets larger and larger. Thus the scattered wave is going to have the following asymptotic form:

$$\psi_{sc} = \psi - e^{ikz} \underset{r\to\infty}{\longrightarrow} f\left(\theta\right)\frac{e^{ikr}}{r} \qquad (9.2)$$

Introduction-f

- What about the function $f(\theta)$?
- This is called *scattering amplitude* and it is the basic measurable physical quantity of our problem since it determines the angular distribution of the scattered beam.
- The angle θ is measured with respect to the direction of the incident beam (taken as the z-axis). The angle ϕ around this axis does not appear in our consideration because we have accepted that the scattering potential does not depend on this angle and thus the problem has a cylindrical symmetry.

The differential scattering cross section-a



Scattering in a certain solid angle

The differential scattering cross section-b

- If we consider the case of the previous figure then the number of particles that arrive at the counter (of cross-section dA and solid angle $d\Omega$) per unit time it will be proportional to the solid angle $d\Omega$ and the intensity of the incident beam which is given by the probability current j_{inc} .

$$\left(dN\right)_{\text{sc}} \propto d\Omega \cdot j_{\text{inc}} \qquad (9.3)$$

- This means that the ratio $\left(dN\right)_{\text{sc}} / d\Omega \cdot j_{\text{inc}}$ will be a characteristic of the scattering potential.

The differential scattering cross section-c

- If we recall that the quantities $\left(dN\right)_{\text{sc}}$, j_{inc} and their units are defined as follows:

$$\left(dN\right)_{\text{sc}} = \frac{\text{number of particles}}{\text{second}} = \text{sec}^{-1} \qquad (9.4)$$

$$j_{\text{inc}} = \frac{\text{number of particles}}{\text{cm}^2 \text{ and per second}} = \text{cm}^{-2} \cdot \text{sec}^{-1} \qquad (9.5)$$

- Then their ratio will have units of area and is called the *differential scattering cross-section*

The differential scattering cross section-d

- The differential scattering cross section is defined as follows:

$$\frac{d\sigma}{d\Omega} = \frac{\left(dN\right)_{sc}}{d\Omega \cdot j_{inc}} \qquad (9.6)$$

- While the total scaterring cross section is given by

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega \qquad (9.7)$$

The physical content of the scattering cross-section

- The total scattering cross-section represents the “area” that the scattering centre “offers” to the incident beam. This means that the part of the beam which is inside a cylinder of area σ will be scattered out of the beam in different directions.

Scattering cross-section and scattering amplitude

- We can show that the differential scattering cross-section and the scattering amplitude are related through a very simple and elegant relation as follows:

$$\frac{d\sigma}{d\Omega} = \left| f\left(\theta\right) \right|^2 \qquad (9.8)$$

The proof is shown in the class

…to be noted

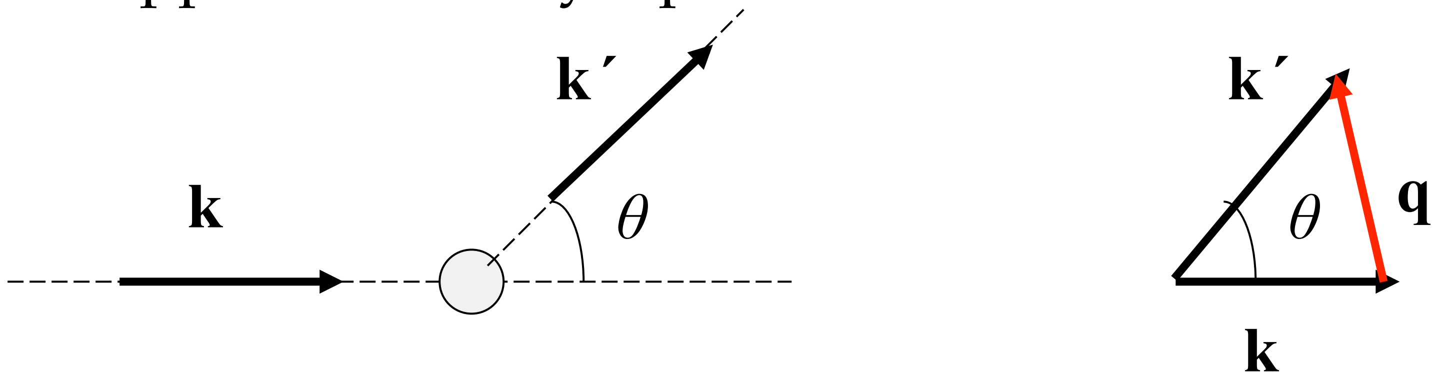
- Most scattering experiments are carried out in the laboratory (lab) frame in which the target is initially at rest while the projectiles are moving.
- Calculations of the cross sections are generally easier to perform within the center-of-mass (CM) frame in which the center of mass of the projectile-target system is at rest (before and after collision). In order to be able to compare the experimental measurements with the theoretical calculations, one has to know how to transform the cross sections from one frame into the other.

…to be noted

- We should note that the total cross section σ is the same in both frames, since the total number of collisions that take place does not depend on the frame in which the observation is carried out.
- On the contrary the differential cross sections $d\sigma(\theta,\phi)/d\Omega$, they are not the same in both frames, since the scattering angles (θ,ϕ) are frame dependent.
- In the case where the interaction between two particles depends only on their relative distance the scattering problem reduces to two decoupled problems: one of the CM which moves like a free particle of mass $M=m_1+m_2$ and which is of no concern to us, and another for a fictitious particle of reduced mass $m=m_1m_2/(m_1+m_2)$ which moves in the inter-particle potential V.

Calculation of scattering amplitude at high energies: The Born approximation

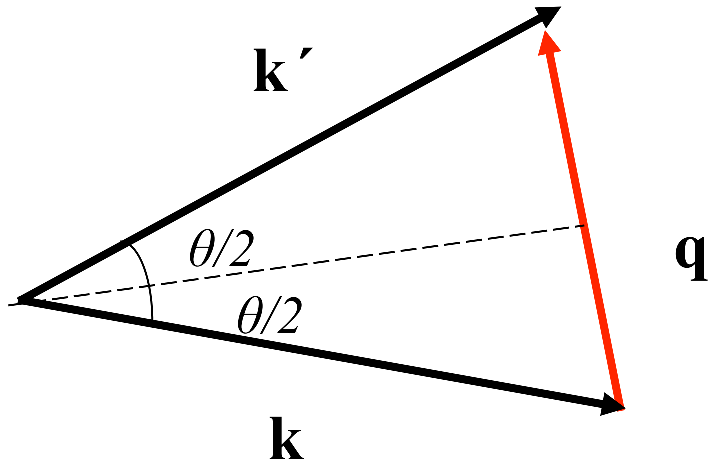
- When the kinetic energy of the incident particles is higher than the scattering potential energy we talk about *weak scattering*. In this case the scattering potential can be treated as small perturbation to the free particle Hamiltonian. The scattered particle is approximated by a plane wave.



The geometry of the vectors **k** and **k'** (initial and final) momentum and the momentum transfer **q**= **k'** -.**k**

Calculation of scattering amplitude at high energies: The Born approximation

In elastic scattering the magnitude of the momentum before the scattering is equal to the magnitude of the momentum after the scattering.



$$|\mathbf{k}| = |\mathbf{k}'|$$

$$q = 2k\sin(\theta / 2)$$

Calculation of scattering amplitude at high energies: The Born approximation

- It can be shown that the scattering amplitude in the Born approximation is given by:

$$f_B(\theta) = -\frac{m}{2\pi\hbar^2}\tilde{V}(\mathbf{q}) \qquad (9.9)$$

$$\tilde{V}(\mathbf{q}) = \int e^{-i\mathbf{q}\cdot\mathbf{r}} V(\mathbf{r}) d^3\mathbf{r} \qquad (9.10)$$

- This is a very simple and physically transparent result. The scattering amplitude is related to the Fourier transform (with respect momentum transfer **q**) of the scattering potential!

Calculation of scattering amplitude at high energies: The Born approximation

- If the scattering potential has a spherical symmetry then it can be shown that:

$$f_B(\theta) = -\frac{2m}{q\hbar^2}\int_0^\infty rV(r)\sin(qr)\,dr \qquad (9.11)$$

$$q = 2k\sin(\theta/2) \qquad (9.12)$$

Examples of the Born approximation: The Yukawa potential

- The Yukawa potential has the form:

$$V(r) = -g\frac{e^{-\lambda r}}{r} \qquad (9.13)$$

- It can be shown that

$$f_B(\theta) = \frac{2mg}{\hbar^2}\frac{1}{\lambda^2 + 4k^2\sin^2\dfrac{\theta}{2}} \qquad (9.14)$$

Detailed discussion will follow in the class

Examples of the Born approximation: The Coulomb potential

- The Coulomb potential has the form:

$$V(r) = -\frac{g}{r} \qquad (9.15)$$

- It can be shown that

$$f_B(\theta) = \frac{g}{4E}\frac{1}{\sin^2\frac{\theta}{2}} \qquad (9.16)$$

- Where E is the kinetic energy of the particle

Detailed discussion will follow in the class

The Born approximation in one dimension: The reflection coefficient

- It can be shown that if we consider the scattering of a beam of particles from a one-dimensional potential $V(x)$ (a potential step or a potential well as you know from basic quantum mechanics) then the reflection coefficient is given by:

$$R(k) = \frac{1}{4k^2}\left|\int_{-\infty}^{\infty} e^{2ikx}U(x)\,dx\right|^2 \qquad (9.17)$$

The Born approximation in one dimension: The reflection coefficient

- In (5.17) the quantity $U(x)$ is given by:

$$U(x) = 2mV(x)/\hbar^2 \qquad (9.18)$$

- The above formula (9.18) is the Born approximation for the reflection coefficient and gives very good results for energies somehow larger than the "height" of scattering potential or the absolute value of the "depth" in the case where the scattering potential is a well

The Born approximation in one dimension: The reflection coefficient

- As you can see from (5.17) the reflection coefficient is the Fourier transform of the potential. In both cases the Fourier variable is the momentum transfer.
- In a one-dimensional problem the momentum transfer can take only two values: a) $q=0$, when the particle goes through the barrier and b) $q=k-(-k)=2k$ in the case of reflection.

Mathematical supplement

- In quantum mechanics the probability current is given by:

$$\mathbf{j} = \mathrm{Re}\left(\psi^*\left(\mathbf{v}\psi\right)\right), \qquad \mathbf{v} = \frac{\mathbf{p}}{m} = -\frac{i\hbar}{m}\vec{\nabla}$$

- In spherical coordinates:

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$