ORTHOGONAL POLYNOMIALS

1) Legendre polyn.

$P_n(x)$

كثيرات الحدود المتعامدة
للوجندر من الدرجة n

a) Differential eq:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$-1 < x < 1$$

b) Rodriguez formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

$$-1 < x < 1$$

Ex: Calculate $P_2(x)$.

Sol:

$$P_2(x) = \frac{1}{8}\left(x^4 - 2x^2 + 1\right)''$$

$$= \frac{1}{8}\left(4x^3 - 4x\right)'$$

$$= \frac{1}{2}\left(x^3 - x\right)'$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$\boxed{P_2(x) = \frac{1}{2}(3x^2 - 1)}$$

c) Orthogonality

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \delta_{nm} \frac{2}{2n+1}$$

where δ_{nm} is the Kronecker symbol

$$\delta_{nm} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

d) Recurrence formula

$$(2n+1)\, x\, P_n(x) = (n+1) P_{n+1}(x) + n\, P_{n-1}(x)$$

Ex: $P_0(x) = 1$;

$P_1(x) = x$

deduce $P_2(x)$

e) First polynomials

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35t^4 - 30t^2 + 3)$$

f) generating function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

2) Hermite polyn.

a) Diff. eq:

$$y'' - 2xy' + 2ny = 0$$

b) Rodriguez formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Ex: Calculate $H_2(x)$ using Rodriguez formula.

Sol:

$$H_2(x) = e^{x^2}\left(e^{-x^2}\right)''$$

$$= e^{x^2}\left(-2x e^{-x^2}\right)'$$

$$= \cancel{e^{x^2}}\left(4x^2 - 2\right)\cancel{e^{-x^2}}$$

$$H_2(x) = 4x^2 - 2$$

c) Orthogonality

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) e^{-x^2} dx = \delta_{nm} \sqrt{\pi}\, 2^n n$$

d) Recurrence formula

$$H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0$$

e) First polyn.

$H_0(x) = 1$; $H_1(x) = 2x$; $H_2(x) = 4x^2 -$

$H_3(x) = 8x^3 - 12x$

f) Generating function

$$e^{-t^2+2xt} = \sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n!}$$

Ex:

a) Calculate $H_1(x)$ using R. f.

b) using $H_1(x)$ and $H_2(x)$ find $H_3(x)$ (rec. f.)

Sol:

$$H_1(x) = -e^{x^2}\frac{d}{dx}\left(e^{-x^2}\right)$$

$$= -\cancel{e^{x^2}}\left(-2x\,\cancel{e^{-x^2}}\right)$$

$$= 2x$$

b) $H_1(x) = 2x$; $H_2(x) = 4x^2 -$

Rec. f: $(n=2)$

$$H_3(x) - 2x\,H_2(x) + 4H_1(x) =$$

So

$$H_3(x) = 2x(4x^2-2) - 4\cdot(2x$$

$$= 8x^3 - 12x$$

Laguerre polyn.

Diff. eq: $(x \geq 0)$

$$x y'' + (1-x) y' + n y = 0$$

Generating function

$$\frac{1}{1-t} e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!}$$

Rodriguez f.

$$L_n(x) = e^x \frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

Ex: calculate $L_1(x)$, $L_2(x)$.
using R. f.

Sol:

$$L_1(x) = e^x \left(x e^{-x}\right)' = \cancel{e^x}(-x+1)\cancel{e^{-x}}$$

$$L_1(x) = 1 - x$$

$$L_2(x) = e^x \left(x^2 e^{-x}\right)'' = e^x\left[(-x^2+2x)e^{-x}\right.$$
$$= x^2 - 2x - 2x + 2$$

$$L_2(x) = x^2 - 4x + 2$$

Orthogonality

$$\int_0^\infty L_n(x) L_m(x) e^{-x} dx = \delta_{nm} (n!)^2$$

Ex: Calculate

$$I = \int_0^\infty (1-x)^2 e^{-x} dx$$

Sol:

$$I = \int_0^\infty L_1(x)^2 e^{-x} dx = (1!)^2 = 1$$

:

Recurrence formula:

$$L_{n+1}(x) - (2n+1-x) L_n(x) + n^2 L_{n-1}(x) = 0$$

Ex: knowing $L_1(x)$ and $L_2(x)$, calculate $L_3(x)$.

Sol: in the rec. f. $n=2$

$$L_3(x) = (5-x)(x^2-4x+2) - 4(1-x)$$

$$L_3(x) = -x^3 + 9x^2 - 18x + 6$$

chap 3)

Fourier series

f periodic f.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots$$

we can write

f(x):

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

where

$$c_{n>0} = \frac{1}{2}(a_n - i b_n)$$

and

$$c_{-n<0} = \frac{1}{2}(a_n + i b_n)$$

$$c_0 = \frac{a_0}{2}$$

We have:

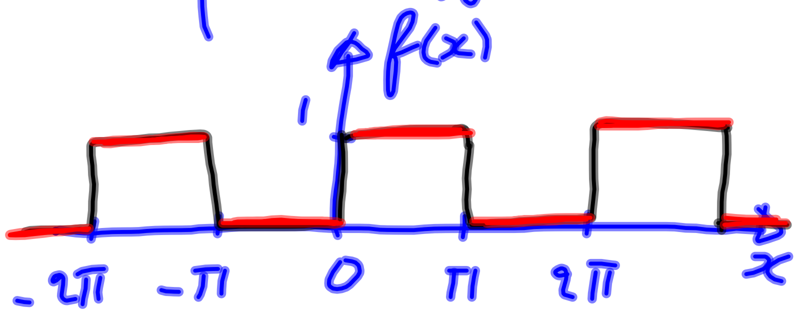
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$\left(a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \right)$$

Ex: Square wave:



$$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases}$$

f is 2π-periodic

Calculate a_0, a_n and b_n $(n \geq 1)$

$$a_0 = \frac{1}{\pi}\int_0^\pi dx = 1$$

$$a_{n\geq 1} = \frac{1}{\pi}\int_0^\pi \cos nx \, dx$$

$$= \frac{1}{\pi}\left[+\frac{\sin nx}{n}\right]_0^\pi$$

$$= \frac{1}{n\pi}(0) = 0$$

$$b_0 = 0$$

$$b_{n\geq 1} = \frac{1}{\pi}\int_0^\pi \sin nx \, dx$$

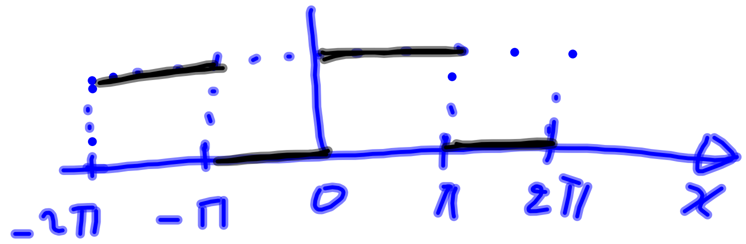
$$= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^\pi$$

$$= \frac{1}{n\pi}(1 - \cos n\pi)$$

$$b_{2p} = 0$$

$$b_{2p+1} = \frac{2}{n\pi} = \frac{2}{(2p+1)\pi}$$

The function is then written as:



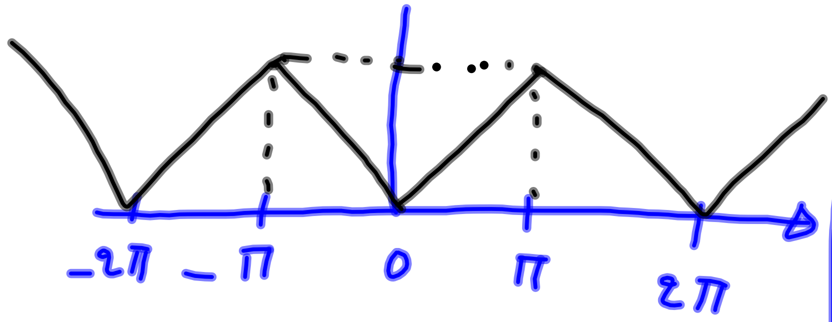
$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$$

for $x = \frac{\pi}{2}$; we find:

$$1 = \frac{1}{2} + \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots\right)$$

or $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$

Triangular wave



f: 2π-periodic

f: even function

$f(x) = x$ if $0 \leq x \leq \pi$

Sol: $b_n = 0$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\,\frac{\pi^2}{2} = \pi$$

and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$u(x) = x \longrightarrow du = dx$$

$$dv(x) = \cos(nx)dx \rightarrow v(x) = \frac{\sin nx}{n}$$

$$a_n = \frac{2}{\pi}\left\{ \cancel{\left[\frac{x\sin nx}{n}\right]_0^\pi} - \int_0^\pi \frac{\sin nx}{n} dx \right\}$$

$$= \frac{2}{n\pi}\left[\frac{\cos nx}{n}\right]_0^\pi = \frac{2\left((-1)^n - 1\right)}{n^2\pi}$$

$$a_{2p} = 0$$

$$a_{2p+1} = -\frac{4}{(2p+1)^2\pi}$$

So:

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{p=0}^{\infty} \frac{\cos(2p+1)x}{(2p+1)^2}$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots \right)$$

We deduce that: $(x=0)$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

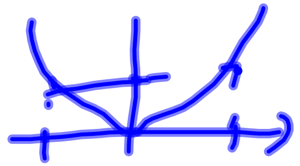
So, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

دالة زوجية if f even

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

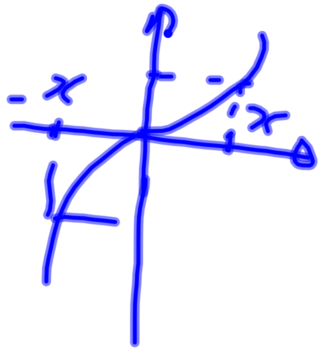
دالة فردية if f odd



$f(x) = f(-x)$

$f(x) = x^2$ even

$f(x) = x$ odd



$f(-x) = -f(x)$

$f(x) = x^2 + x$ is not even and not odd.

Ex:
f 2π-periodic and

$f(x) = x$ for $-\pi < x < \pi$

Find the Fourier coef.

f is odd

So $a_0 = a_n = 0$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$u = x \qquad dv = \sin nx\,dx$

$du = dx \qquad v = -\frac{\cos nx}{n}$

$$b_n = \frac{2}{\pi}\left(-\left[\frac{x\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right)$$

$$= \frac{2}{\pi}\left(-\frac{\pi(-1)^n}{n} + \cancel{\frac{1}{n^2}\left[\sin nx\right]_0^{\pi}}\right)$$

$$= \frac{2}{\pi}\quad\frac{\pi(-1)^{n+1}}{n} = \frac{2(-1)^{n+1}}{n}$$

and

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

Parseval's identity

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Ex: we find that $f(x) = 1 = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$

for $0 < x < \pi$

So

$$1 = \frac{1}{\pi}\int_0^{\pi} 1^2\,dx = \frac{1}{2} + \sum_{p=1}^{\infty}\frac{16}{(2p+1)^2\pi^2}$$

(Chap 4)

Fourier Transforms

The Fourier transform of a function $f(t)$ is:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{i\omega t} \, dt$$

$$f(t) \xrightarrow{F.T} F(\omega)$$

The inverse relation will be:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) e^{-i\omega t} \, d\omega$$

we have

$$F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \, dt$$

$$f(t) \xrightarrow{F.T} F.T[f(t)] = F(\omega)$$

$$\frac{df}{dt} \xrightarrow{F.T} F.T\left[\frac{df}{dt}\right] = G(\omega)$$

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{df}{dt} e^{i\omega t} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \, i\omega e^{i\omega t} \, dt$$

$$G(\omega) = i\omega F(\omega)$$

$$f(t) \xrightarrow{F.T} F(\omega)$$

$$\frac{df}{dt} = f'(t) \xrightarrow{F.T} i\omega F(\omega$$

$$\frac{d^2f}{dt^2} = f''(t) \xrightarrow{F.T} -\omega^2 F(\omega$$

Convolution of 2 function $f(t)$ and $g(t)$

$$h(t) = (f * g)(t)$$

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(u) f(t-u)\, du$$

$h = f * g$ = convolution of f and g.

we have $f * g = g * f$

we have:

Th:

$$\boxed{FT[f*g] = FT[f]\cdot FT[g]}$$

because:

$$e^{i\omega t} = e^{i\omega u}\cdot e^{i\omega(t-u)}$$

Prove:

$$FT[f*g] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} (f*g)(t)\, e^{i\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(u) f(t-u)\, e^{i\omega t}\, du\, dt$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(u)\, e^{i\omega u}\, du}_{FT(g)} \cdot \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t-u)\, e^{i\omega(t-u)}\, d(t-u)}_{F.T[f]}$$

we use F.T to find the convolution of two functions f and g.

$$f*g = FT^{-1}\left[FT(f).FT(g)\right]$$

HW: f 2π-periodic

$$f(x) = \begin{cases} x & \text{if } 0<x<\pi \\ -x & \text{if } -\pi<x<0 \end{cases}$$

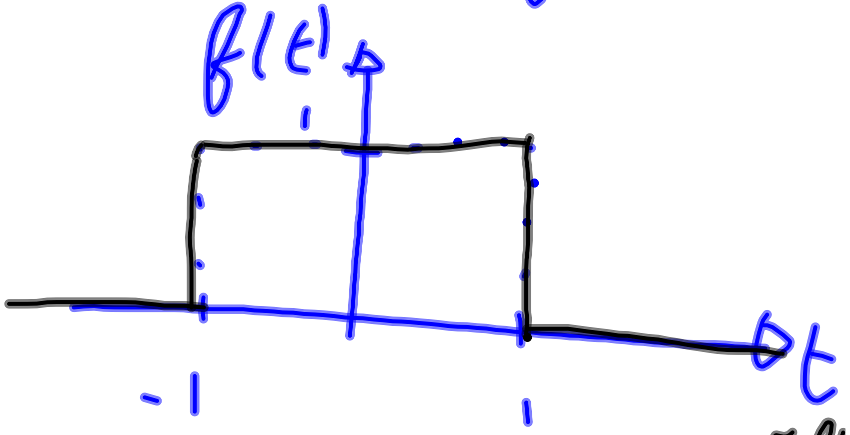
1) Represent $f(x)$ by a Fourier series.

2) Show that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{8}$$

Ex: Find the F.T of

$$f(t) = \begin{cases} 1 & \text{if } |t|<1 \\ 0 & \text{if } |t|>1 \end{cases}$$



$$F(w) = FT[f($$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{i\omega t}\, dt$$

with $f(t) = 1$ if $-1 < t < 1$

$= 0$ if $|t| > 1$

So

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\omega t}}{i\omega} \right]_{-1}^{1}$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\omega} - e^{-i\omega}}{i\omega} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2 \sin\omega}{\omega}$$

$$\boxed{F(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}}$$

chap 5

Laplace transform

E : set of Laplace.

$f \in E$ if:

1- $f(t<0) = 0$

2- $f(t>0)$ cont.

3. it exists a as $f(t)e^{-at}$ is limited (majored)

$$F(p) = \int_0^\infty f(t) e^{-pt} dt$$

$$f(t) \xrightarrow{L.T} F(p)$$

Ex:

$f(t) = 1 \longrightarrow F(p) = \frac{1}{p}$

because:

$$F(p) = \int_0^\infty e^{-pt} dt = \left[\frac{e^{-pt}}{-p}\right]_0^\infty$$

$$= \frac{1}{p}$$

$f(t) = t^n \xrightarrow{L.T} F(p)\,?$

$$F(p) = \int_0^\infty t^n e^{-pt}\,dt$$

we put $u = pt$

$du = p\,dt$

$$F(p) = \int_0^\infty \frac{u^n}{p^n} e^{-u} \frac{du}{p}$$

$$= \frac{1}{p^{n+1}} \Gamma(n+1) = \frac{n!}{p^{n+1}}$$

$$t^n \xrightarrow{L.T} \frac{n!}{p^{n+1}}$$

$$1 \longrightarrow \frac{1}{p}$$

$$t \longrightarrow \frac{1}{p^2}$$

$$t^2 \longrightarrow \frac{2}{p^3}$$

$$t^3 \longrightarrow \frac{6}{p^4}$$

Laplace transform of derivatives

$$f(t) \longrightarrow F(p)$$

$$f'(t) \longrightarrow G(p)$$

$$G(p) = \int_0^\infty f'(t) \underbrace{e^{-pt}}_{g(t)} dt$$

$$= \left[f(t) e^{-pt} \right]_0^\infty + p \int_0^\infty f(t) e^{-pt} dt$$

$$G(p) = -f(0) + pF(p)$$

So:

$$f(t) \xrightarrow{L.T} F(p)$$

$$f'(t) \xrightarrow{L.T} pF(p) - f(0)$$

Ex: $y(t) \xrightarrow{L.T} Y(p)$

$$y' \xrightarrow{L.T} pY(p) - y(0)$$

$$y'' \xrightarrow{L.T} ?$$

$$y(t) \xrightarrow{L.T} Y(p)$$

$$y' \xrightarrow{L.T} pY(p) - y(0)$$

$$y'' \xrightarrow{L.T} p\left(pY(p) - y(0)\right) - y'(0) = p^2 Y(p) - py(0) - y'(0)$$

$$L[e^{\lambda t}] \stackrel{?}{=} \frac{1}{p-\lambda}$$

$$f(t) = e^{\lambda t}; \; F(p) = ?$$

$$F(p) = \int_0^\infty e^{\lambda t} e^{-pt} \, dt$$

$$= \int_0^\infty e^{-(p-\lambda)t} \, dt$$

converge only if $\Re(p) > \Re(\lambda)$

$$F(p) = \left[\frac{e^{-(p-\lambda)t}}{-(p-\lambda)} \right]_0^\infty$$

$$= \frac{1}{p-\lambda}$$

$$t^\alpha \xrightarrow{L.T.} \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$$

$f(t) = t^{\alpha}$

$$F(p) = \int_0^{\infty} t^{\alpha} e^{-pt}\, dt$$

$u = pt$ so:

$$F(p) = \int_0^{\infty} \frac{u^{\alpha}}{p^{\alpha}} e^{-u} \frac{du}{p}$$

$$= \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$$

Ex: we have

$$L.T[e^{\lambda t}] = \frac{1}{p-\lambda}$$

deduce $L.T[\sin\omega t]$

and $L.T[\cos\omega t]$

$$\left(e^{i\omega t} = \cos\omega t + i\sin\omega t\right)$$

we have:

$$L.T[e^{i\omega t}] = \frac{1}{p-i\omega}$$

$$= \frac{p+i\omega}{p^2+\omega^2}$$

$$LT[e^{i\omega t}] = \frac{p}{p^2+\omega^2} + i\frac{\omega}{p^2+\omega^2}$$

$$LT[\cos\omega t + i\sin\omega t]$$

So

$$L.T[\cos\omega t] = \frac{p}{p^2+\omega^2}$$

and

$$L.T[\sin\omega t] = \frac{\omega}{p^2+\omega^2}$$

Ex: Calculate:

$$L.T[\sqrt{t}]$$

$$LT[\sqrt{t}] = \int_0^\infty t^{\frac{1}{2}} e^{-pt}\,dt$$

we put: $u = pt$, so

$$L.T[\sqrt{t}] = \int_0^\infty \frac{u^{\frac{1}{2}}}{p^{\frac{1}{2}}} e^{-u} \frac{du}{p}$$

$$= \frac{1 \times \Gamma(\frac{3}{2})}{p^{3/2}} = \frac{\Gamma(\frac{3}{2})}{p^{3/2}}$$

$$LT[\sqrt{t}] = \frac{\Gamma(3/2)}{p^{3/2}}$$

$$= \frac{1}{2}\frac{\Gamma(\frac{1}{2})}{p^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2\,p^{3/2}}$$

Ex: Calculate:

$$L.T[t e^{\lambda t}]$$

$$L.T[t e^{\lambda t}] = \int_0^\infty t\, e^{-(p-\lambda)t}\,dt$$

$$u = t \rightarrow du = dt$$

$$dv = e^{-(p-\lambda)t}dt \rightarrow v = \frac{e^{-(p-\lambda)t}}{-(p-\lambda)}$$

$$L.T[t e^{\lambda t}] = \cancel{\left[\frac{t\, e^{-(p-\lambda)t}}{-(p-\lambda)}\right]_0^\infty}$$

$$+\int_0^\infty \frac{e^{-(p-\lambda)t}}{p-\lambda}\,dt = \frac{1}{(p-\lambda)^2}$$

Ex:
Prove that
$$LT[t f(t)] = -\frac{dF}{dp}$$

Sol:
we have:
$$F(p) = \int_0^\infty f(t)\, e^{-pt}\, dt$$

So
$$\frac{dF}{dp} = -\int_0^\infty (t f(t))\, e^{-pt}\, dt$$
$$= -LT[t f(t)]$$

Appl:
$$* \; LT[t\, e^{\lambda t}] = ?$$

Sol:
we have:
$$LT[e^{\lambda t}] = \frac{1}{p-\lambda}$$

So $LT[t\, e^{\lambda t}] = -\frac{d}{dp}\left(\frac{1}{p-\lambda}\right)$
$$= \frac{1}{(p-\lambda)^2}$$

* $LT[t\sin\omega t] = ?$

Sol:
we have

$$LT[\sin\omega t] = \frac{\omega}{p^2+\omega^2}$$

So

$$LT[t\sin\omega t] = -\frac{d}{dp}\left[\frac{\omega}{p^2+\omega^2}\right]$$

$$= + \frac{2\omega p}{(p^2+\omega^2)^2}$$

Ex: Find:

$$LT^{-1}\left[\frac{1}{(p+3)(p-1)}\right]$$

Sol:

$$\frac{1}{(p+3)(p-1)} = \frac{a}{p+3} + \frac{b}{p-1}$$

$$= \frac{a(p-1)+b(p+3)}{(p+3)(p-1)}$$

$$\underbrace{p(a+b)}_{=0} + \underbrace{3b-a}_{=1} = 1$$

$a = -b$

$3b - a = 3b + b = 1$

$\Rightarrow b = \frac{1}{4}$ and $a = -\frac{1}{4}$

So

$$\frac{1}{(p+3)(p-1)} = -\frac{1}{4}\frac{1}{p+3} + \frac{1}{4}\frac{1}{p-1}$$

$$= \frac{1}{4}\left[\frac{1}{p-1} - \frac{1}{p+3}\right]$$

$$LT^{-1}\left[\frac{1}{(p+3)(p-1)}\right]$$

$$= \frac{1}{4} LT^{-1}\left[\frac{1}{p-1} - \frac{1}{p+3}\right]$$

$$= \frac{1}{4}\left\{LT^{-1}\left[\frac{1}{p-1}\right] - LT^{-1}\left[\frac{1}{p+3}\right]\right\}$$

$$= \frac{1}{4}\left(e^{t} - e^{-3t}\right)$$

Ex: Solve:

$$y'' - 3y' + 2y = 4e^{2t}$$

with $y(0) = -3$ and $y'(0) = -5$

Sol:

$$LT[y(t)] = Y(p)$$

$$LT[y'] = pY(p) + 3$$

$$LT[y''] = p^2 Y(p) + 3p - 5$$

$$LT[e^{2t}] = \frac{1}{p-2}$$

$$p^2 Y(p) + 3p - 5 - 3\left(pY(p) + 3\right) + 2Y(p) = \frac{4}{p-2}$$

$$(p^2 - 3p + 2)Y(p) = 14 + \frac{4}{p-2} - 3p$$

$$Y(p) = \frac{-3p^2 + 20p - 24}{(p-2)^2(p-1)}$$

$$= \frac{a}{p-1} + \frac{b}{p-2} + \frac{c}{(p-2)^2}$$

$$= \frac{a(p-2)^2 + b(p-1)(p-2) + c(p-}{(p-1)(p-2)^2}$$

by identificatio

$\Rightarrow a = 7$

$b = 4$

$c = 4$

$$Y(p) = -\frac{7}{p-1} + \frac{4}{p-2} + \frac{4}{(p-2)^2}$$

By LT^{-1}, we obtain:

$$y(t) = -7e^{t} + 4e^{2t} + 4te^{2t}$$