GENERAL MATHEMATICS 2

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Main Contents

Definition of Differential Equations

1. A differential equation is an equation which contains derivatives of the unknown.
2. There are two classes of the differential equations:
   - Ordinary Differential Equations (O.D.E.)
   - Partial Differential Equations (P.D.E.).

We only consider the ordinary differential equations.

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An equation that involves $x, y, y', y'', y''', y^{(4)}, \ldots, y^{(n)}$ for a function $y(x)$ with n^{th} derivative of y with respect to x is an ordinary differential equation of order n.

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Example

1. $y' = x^2 + 5$ *is a differential equation of order 1.*
2. $y'' + x(y')^3 - y = x$ *is a differential equation of order 2.*
3. $(y^{(4)})^3 + x^2 y'' = 2x$ *is a differential equation of order 4.*

Definition of Differential Equations

Notes:

1. $y = y(x)$ is called a solution of a differential equation if it satisfies that differential equation.
2. $y = y(x) + c$ is the general solution of the differential equation.
3. If an initial condition was added to the differential equation to assign a certain value for c, then $y = y(x)$ is called the particular solution of the differential equation.

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Example

Verify that $y = 4x^3 + 2x^2 + x$ *is a solution of the differential equation* $y' = 12x^2 + 4x + 1$. *Then, with the initial condition* $y(0) = 2$, *find the particular solution of the equation.*

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Solution: **(1) The general solution:**

$$y = 4x^3 + 2x^2 + x \Rightarrow y' = 12x^2 + 4x + 1$$

This is the given differential equation, so $y = 4x^3 + 2x^2 + x + c$ is the general solution of the given differential equation.

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Solution: **(1) The general solution:**

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$$y(0) = 2 \Rightarrow 4(0)^3 + 2(0)^2 + 0 + c = 2 \Rightarrow c = 2 \ .$$

Therefore, $y = 4x^3 + 2x^2 + x + 2$ is the particular solution of the differential equation $y' = 12x^2 + 4x + 1$.

Separable Differential Equations

A differential equation is separable if the equation can be written in one of the following forms:

$$M(x) + N(y)y' = 0 \quad \text{OR} \quad M(x) + N(y)\frac{dy}{dx} = 0$$

where $M(x)$ and $N(y)$ are continuous functions and $y' = \frac{dy}{dx}$.

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To solve the separable differential equation, we have the following steps:

1. Write the equation as $M(x)dx + N(y)dy = 0$. This implies $N(y)dy = -M(x)dx$.
2. Integrate the left-hand side with respect to y and the right-hand side with respect to x: $\displaystyle\int N(y)dy = \int -M(x)dx$.
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Example

Solve the differential equation $y' - y^2e^x = 0$.

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Solution: Manipulate the differential equation to become $N(y)dy = -M(x)dx$.

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Example

Solve the differential equation $y' - y^2e^x = 0$.

Solution: Manipulate the differential equation to become $N(y)dy = -M(x)dx$.

$$y' - y^2e^x = 0 \Rightarrow \frac{dy}{dx} = y^2e^x \Rightarrow \frac{dy}{y^2} = e^x\ dx$$

$$\Rightarrow \int y^{-2}\ dy = \int e^x\ dx \qquad \text{integrate both sides}$$

$$\Rightarrow \frac{y^{-1}}{-1} = e^x + c \Rightarrow \frac{1}{y} = -(e^x + c)$$

$$\Rightarrow y(x) = -\frac{1}{e^x + c}. \qquad \text{solve for } y$$

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Example

Solve the differential equation $\frac{dy}{dx} = y\ x$, *with* $y(0) = 1$.

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Solution: **(1) The general solution:**

Separable Differential Equations

Example

Solve the differential equation $\frac{dy}{dx} = y\ x$*, with* $y(0) = 1$*.*

Solution: **(1) The general solution:** Write the differential equation in the form $N(y)dy = -M(x)dx$.

$$\frac{dy}{dx} = y\ x \Rightarrow \frac{dy}{y} = x\ dx$$

$$\Rightarrow \int \frac{1}{y}\ dy = \int x\ dx \qquad \text{integrate both sides} \quad \text{Remember: } \int \frac{1}{x}\ dx = \ln|x| + c$$

$$\Rightarrow \ln|y| = \frac{x^2}{2} + c$$

$$\Rightarrow e^{\ln|y|} = e^{\frac{x^2}{2}+c} \qquad \text{solve for } y \text{ by taking } e \text{ for both sides}$$

$$\Rightarrow y(x) = e^{\frac{x^2}{2}+c} \qquad \text{Remember: } e^{\ln|y|} = y$$

Example

Solve the differential equation $\frac{dy}{dx} = y\ x$, *with* $y(0) = 1$.

Solution: **(1) The general solution:** Write the differential equation in the form $N(y)dy = -M(x)dx$.

$$\frac{dy}{dx} = y\ x \Rightarrow \frac{dy}{y} = x\ dx$$

$$\Rightarrow \int \frac{1}{y}\ dy = \int x\ dx$$ integrate both sides Remember: $\int \frac{1}{x}\ dx = \ln|x| + c$

$$\Rightarrow \ln|y| = \frac{x^2}{2} + c$$

$$\Rightarrow e^{\ln|y|} = e^{\frac{x^2}{2}+c}$$ solve for y by taking e for both sides

$$\Rightarrow y(x) = e^{\frac{x^2}{2}+c}$$ Remember: $e^{\ln|y|} = y$

(2) The particular solution:

$$y(0) = 1 \Rightarrow e^{c} = 1 \Rightarrow \ln(e^{c}) = \ln(1)$$ solve for c by taking ln for both sides

$$\Rightarrow c\ \ln(e) = 0$$ Remember: $\ln x^r = r\ \ln x$ and $\ln(1) = 0$

$$\Rightarrow c\ (1) = 0$$ Remember: $\ln e = 1$

$$\Rightarrow c = 0$$

The particular solution is $y(x) = e^{\frac{x^2}{2}}$.

Example

Solve the differential equation $dy - (1 + y^2) \sin x \; dx = 0$.

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Example

Solve the differential equation $dy - (1 + y^2)\sin x\ dx = 0$.

Solution: Write the differential equation in the form $N(y)dy = -M(x)dx$.

$$dy - (1 + y^2)\sin x\ dx = 0 \Rightarrow dy = (1 + y^2)\sin x\ dx$$

$$\Rightarrow \frac{dy}{1 + y^2} = \sin x\ dx$$

$$\Rightarrow \int \frac{1}{1 + y^2}\ dy = \int \sin x\ dx$$

integrate both sides

$$\Rightarrow \tan^{-1} y = -\cos x + c$$

Remember: $\int \frac{1}{1 + x^2}\ dx = \tan^{-1} x + c$

$$\Rightarrow y(x) = \tan(-\cos x + c)\ .$$

solve for y by taking tan function for both sides