Lecture 2: Review of Vector Calculus

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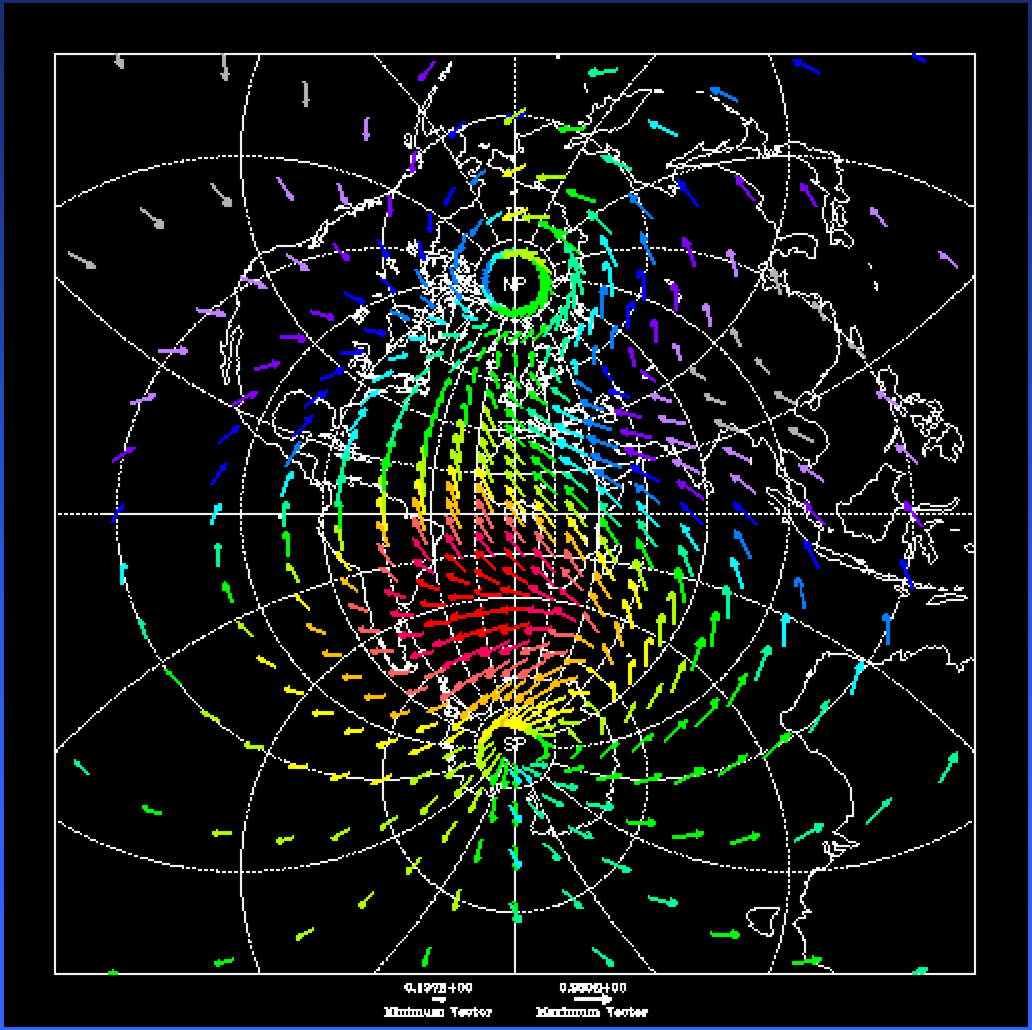


Vector norm

For an n-dimensional vector $x = [x_1 \quad x_2 \quad \dots \quad x_n]$

the vector norm: $|x|_p \equiv \|x\|_p \square \left(\sum_i |x_i|^p \right)^{1/p} \; ; \; p = 1, 2, \dots$ (2.2.1)

$Special\ case: |x|_\infty \square \max_i |x_i|$ (2.2.2)

$Most\ commonly\ used\ L^2 - norm: |x|_2 = |x| = \sqrt{x_1^2 + x_2^2 + \dots x_n^2}$ (2.2.3)

Example: v = (1, 2, 3)

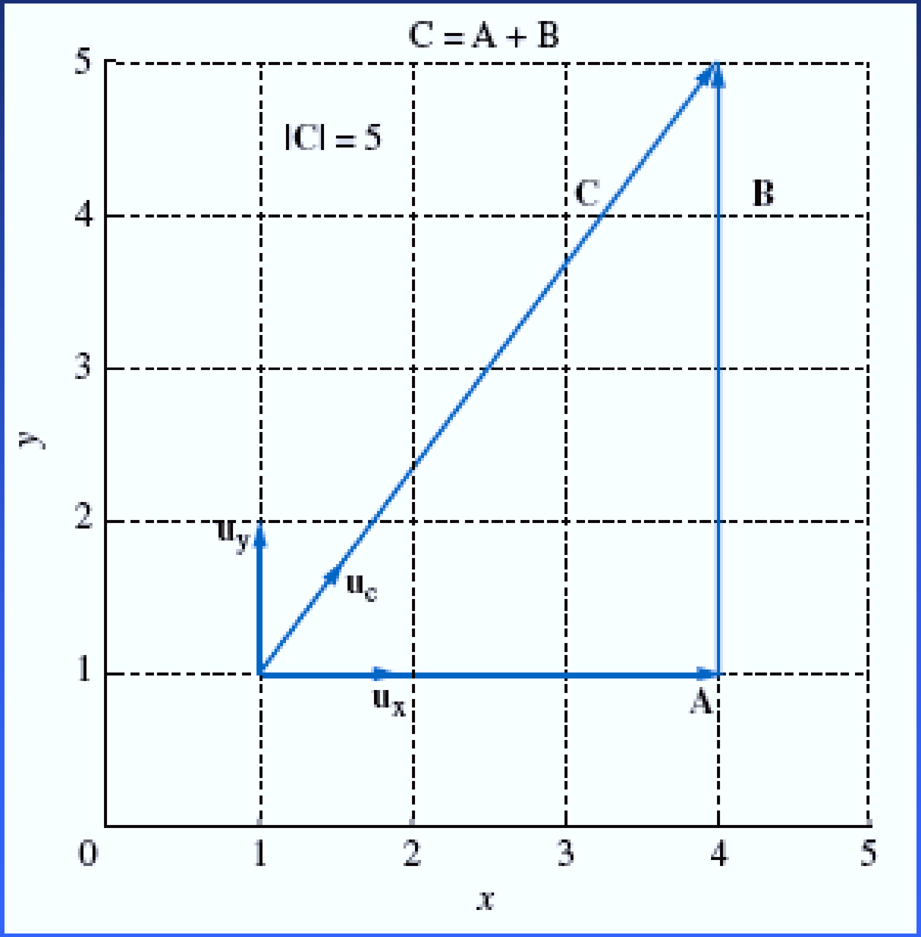
Name	Symbol	value
L^1 – norm	$\lvert v\rvert_1$	6
L^2 – norm	$\lvert v\rvert_2$	$14^{1/2} \cong 3.74$
L^3 – norm	$\lvert v\rvert_3$	$6^{2/3} \cong 3.3$
L^4 – norm	$\lvert v\rvert_4$	$2^{1/4} 7^{1/2} \cong 3.15$
L^∞ – norm	$\lvert v\rvert_\infty$	3

<u>Properties</u>:

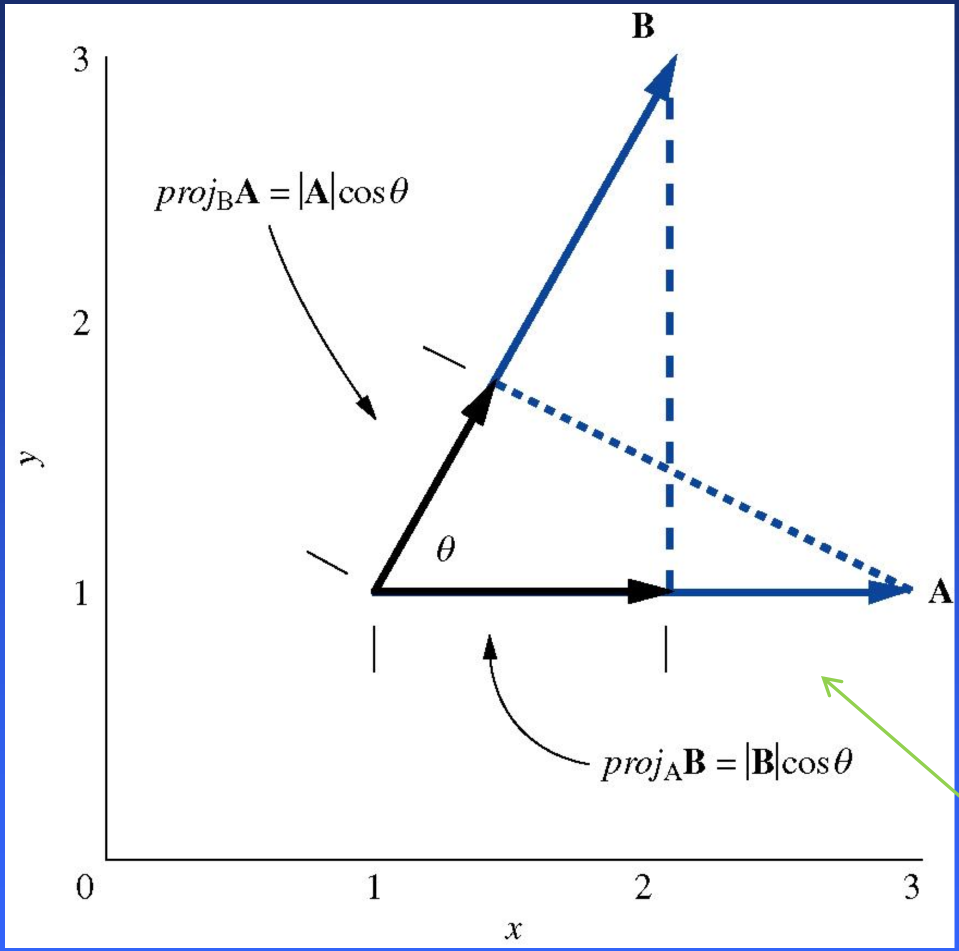
1. $|x| > 0 \;\; when \;\; x \neq 0; \; |x| = 0 \;\; iff \;\; x = 0$ (2.2.4)
2. $|kx| = |k||x| \;\; \forall \; scalar \; k$ (2.2.5)
3. $|x + y| \leq |x| + |y|$ (2.2.6)

Norm(x,p)

Vector sum



Scalar (dot) product



Definitions:

$$\boxed{A \cdot B \equiv AB\cos\theta} \qquad (2.4.1)$$

$$A \cdot B \equiv |A||B|\cos\theta \qquad (2.4.2)$$

$$A \cdot B \equiv A_x B_x + A_y B_y + A_z B_z \qquad (2.4.3)$$

Property: $A \cdot B = B \cdot A$ (2.4.4)

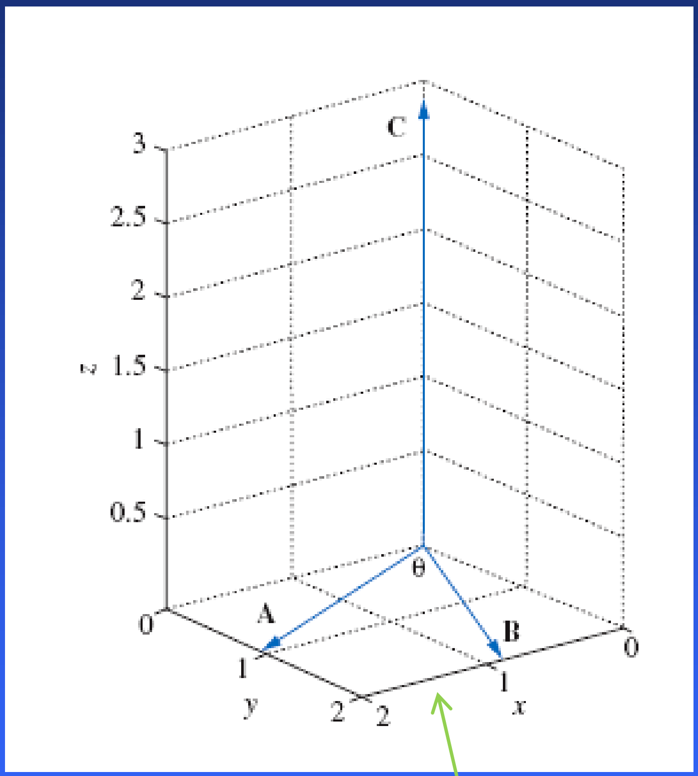
Scalar projection:

$$proj_B A = \frac{A \cdot B}{|B|} = |A|\cos\theta \qquad (2.4.5)$$

$$\begin{cases} A = 2u_x;\ \ B = u_x + 2u_y \\ \Rightarrow \ \ A \cdot B = 2 \end{cases} \qquad (2.4.6)$$

dot(A,B)

Vector (cross) product



Definitions:

$$\boxed{A \times B \equiv AB \sin\theta u_{A\times B}} \quad (2.5.1)$$

Properties: $A \times B = -B \times A$ (2.5.2)

$$A \times B = 0 \Rightarrow A \| B \quad (2.5.3)$$

In the Cartesian coordinate system:

$$A \times B = \begin{vmatrix} u_x & u_y & u_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \quad (2.5.4)$$

$$(A_y B_z - A_z B_y) u_x + (A_z B_x - A_x B_z) u_y + (A_x B_y - A_y B_x) u_z$$

$$\left\{\begin{array}{l} A = 2u_x + u_y; \; B = u_x + 2u_y \\ \Rightarrow C = A \times B = 3u_z \end{array}\right. \Rightarrow A \perp C \; and \; B \perp C \quad (2.5.5)$$

cross(A,B)

Triple products

1. Scalar triple product:

$$A\cdot\left(B\times C\right)=B\cdot\left(C\times A\right)=C\cdot\left(A\times B\right) \tag{2.6.1}$$

2. Vector triple product:

$$A\times\left(B\times C\right)=B\left(A\cdot C\right)-C\left(A\cdot B\right) \tag{2.6.2}$$

Note: (2.6.1) represents a circular permutation of vectors.

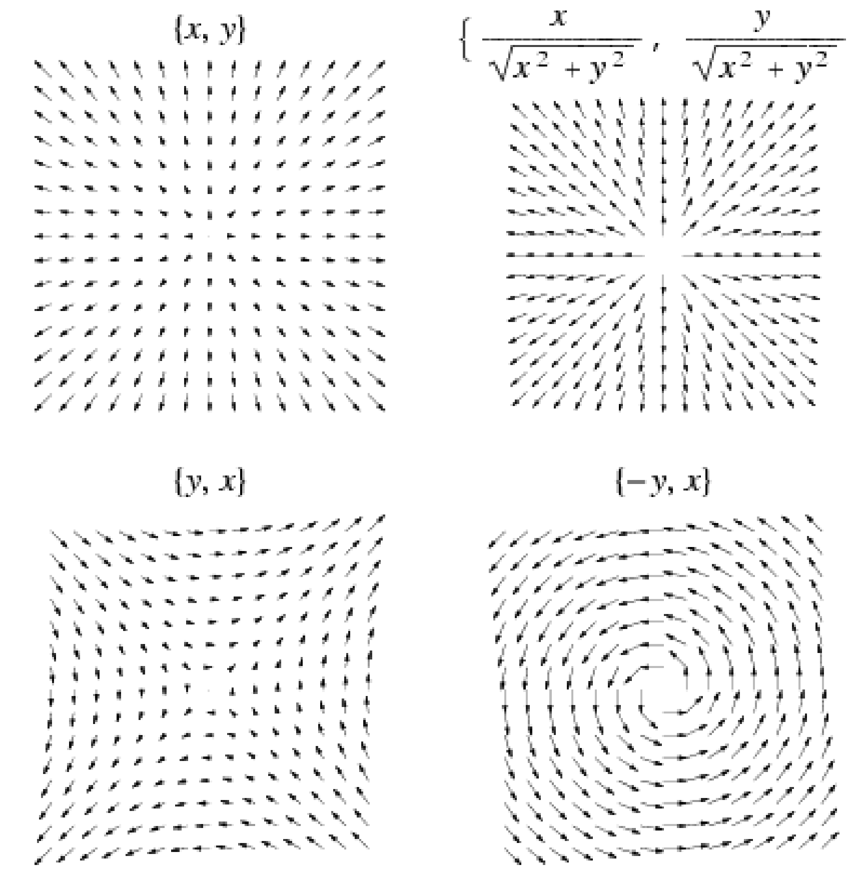
Q: A result of a dot product is a scalar, a result of a vector product is a vector. What is about triple products?

Vector fields

A vector field is a map ***f*** that assigns each vector ***x*** a vector function ***f(x)***.

A vector field is a construction, which associates a vector to every point in a (locally) Euclidean space.

A vector field is uniquely specified by giving its divergence and curl within a region and its normal component over the boundary.



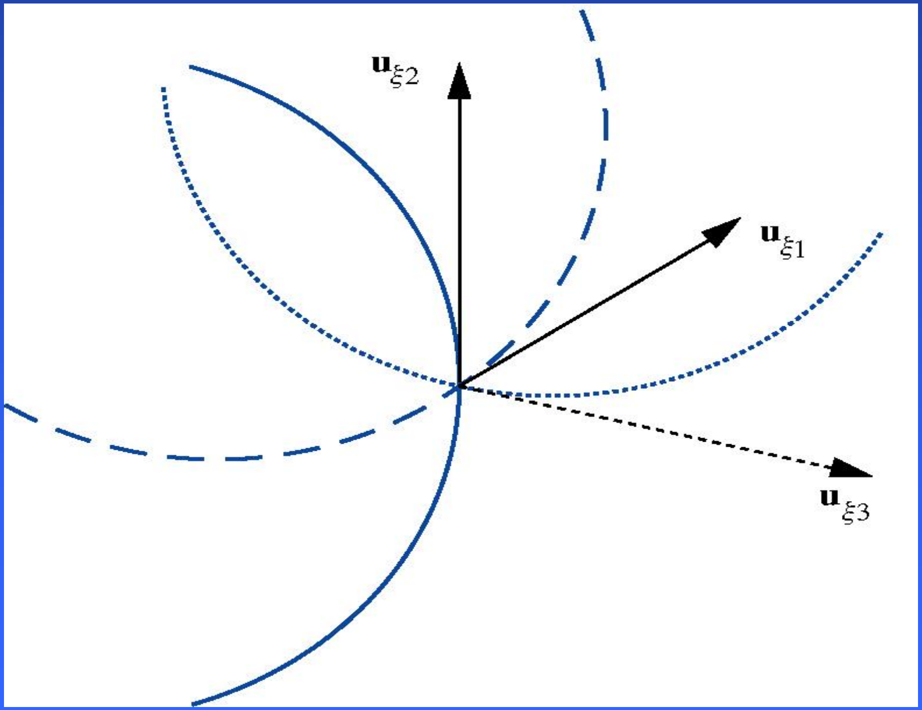
From Wolfram MathWorld

Coordinate systems

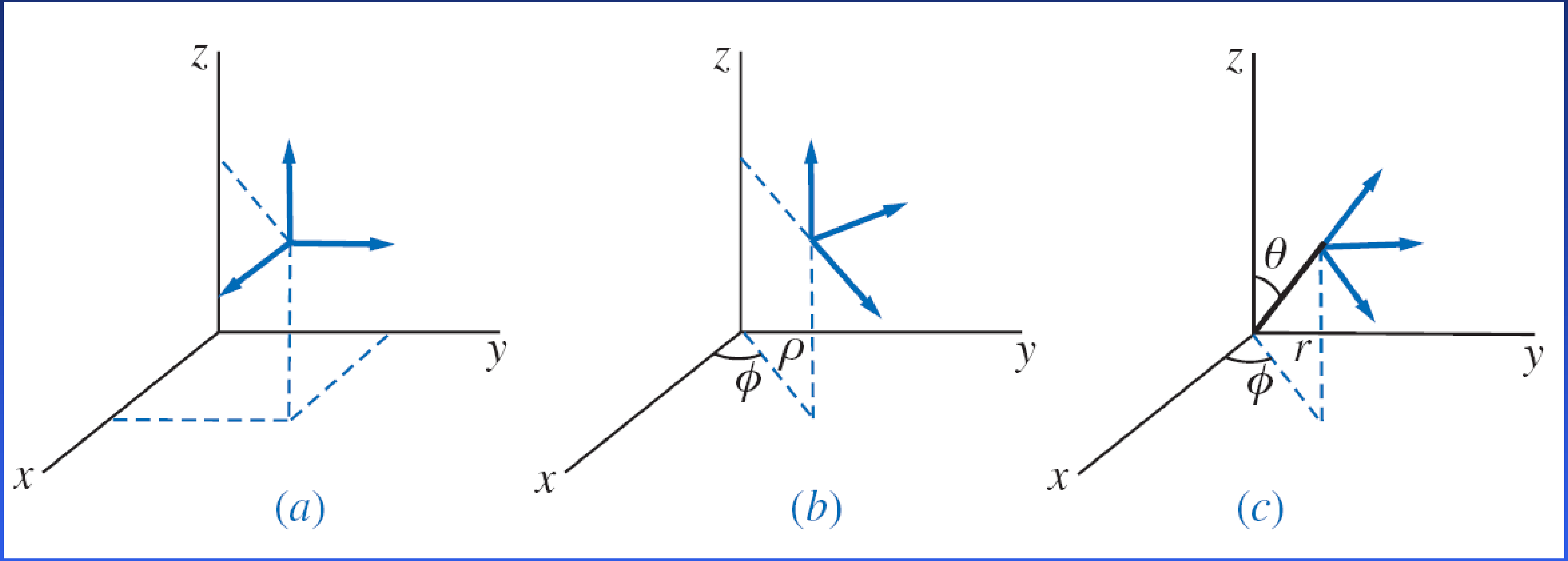
• In a 3D space, a coordinate system can be specified by the intersection of 3 surfaces.
• An orthogonal coordinate system is defined when these three surfaces are mutually orthogonal at a point.

The cross-product of two unit vectors defines a unit surface, whose unit vector is the third unit vector.

A general orthogonal coordinate system: the unit vectors are mutually orthogonal



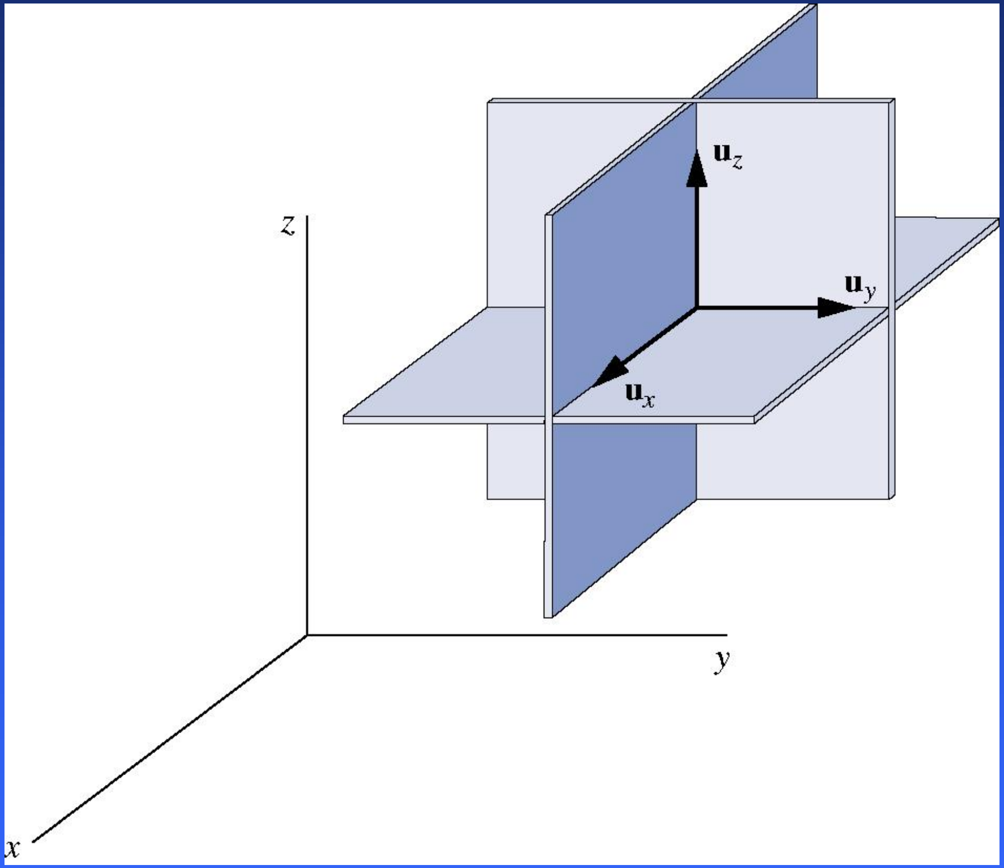
Most commonly used coordinate systems



(a) – Cartesian; (b) – Cylindrical; (c) – Spherical.

In Cartesian CS, directions of unit vectors are independent of their positions;
In Cylindrical and Spherical systems, directions of unit vectors depend on positions.

Coordinate systems: Cartesian



An intersection of 3 planes:
x = const; y = const; z = const

Properties:

$$u_x \cdot u_x = u_y \cdot u_y = u_z \cdot u_z = 1; \quad (2.10.1)$$

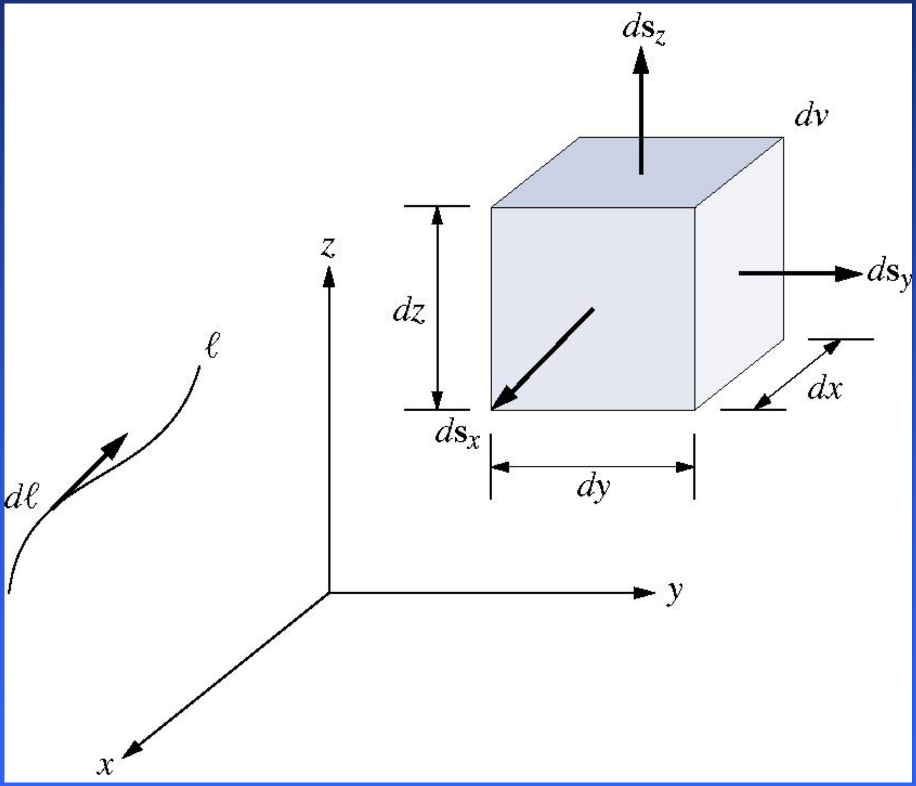
$$u_x \cdot u_y = u_x \cdot u_z = u_y \cdot u_z = 0. \quad (2.10.2)$$

$$\left.\begin{aligned} u_x \times u_y &= u_z \\ u_y \times u_z &= u_x \\ u_z \times u_x &= u_y \end{aligned}\right\} \quad (2.10.3)$$

An arbitrary vector:

$$A = A_x u_x + A_y u_y + A_z u_z \quad (2.10.4)$$

Coordinate systems: Cartesian



A differential line element:

$dl = u_x\, dx + u_y\, dy + u_z\, dz$ (2.11.1)

Three of six differential surface elements:

$ds_x = u_x\, dydz$

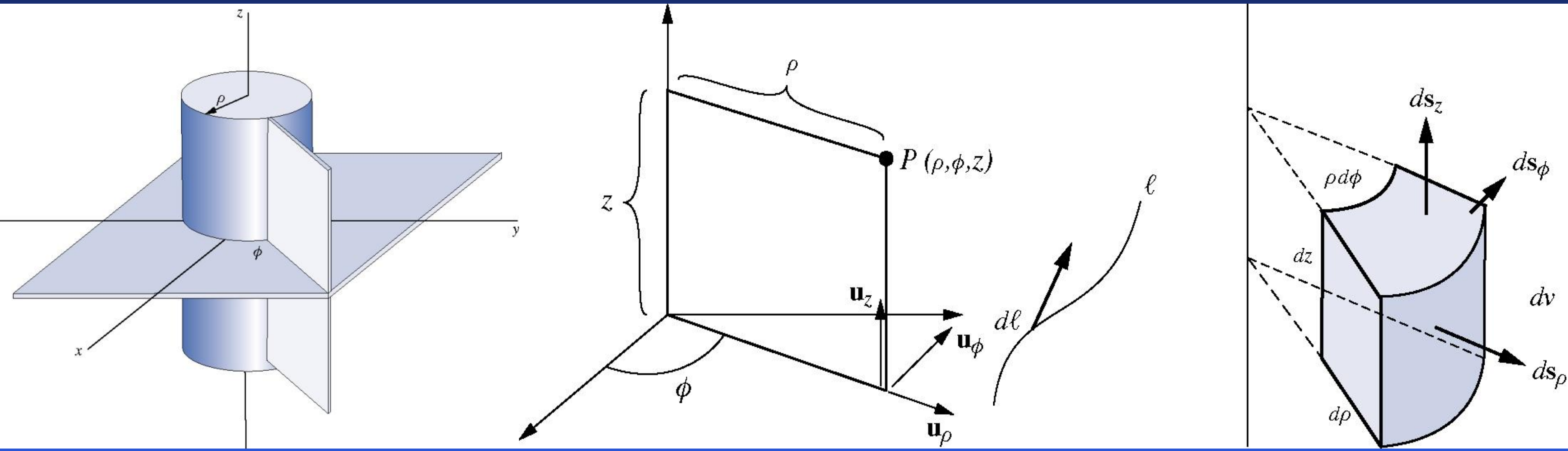
$ds_y = u_y\, dxdz$ (2.11.2)

$ds_z = u_z\, dxdy$

The differential volume element

$dv = dxdydz$ (2.11.3)

Coordinate systems: Cylindrical (polar)



An intersection of a cylinder and 2 planes

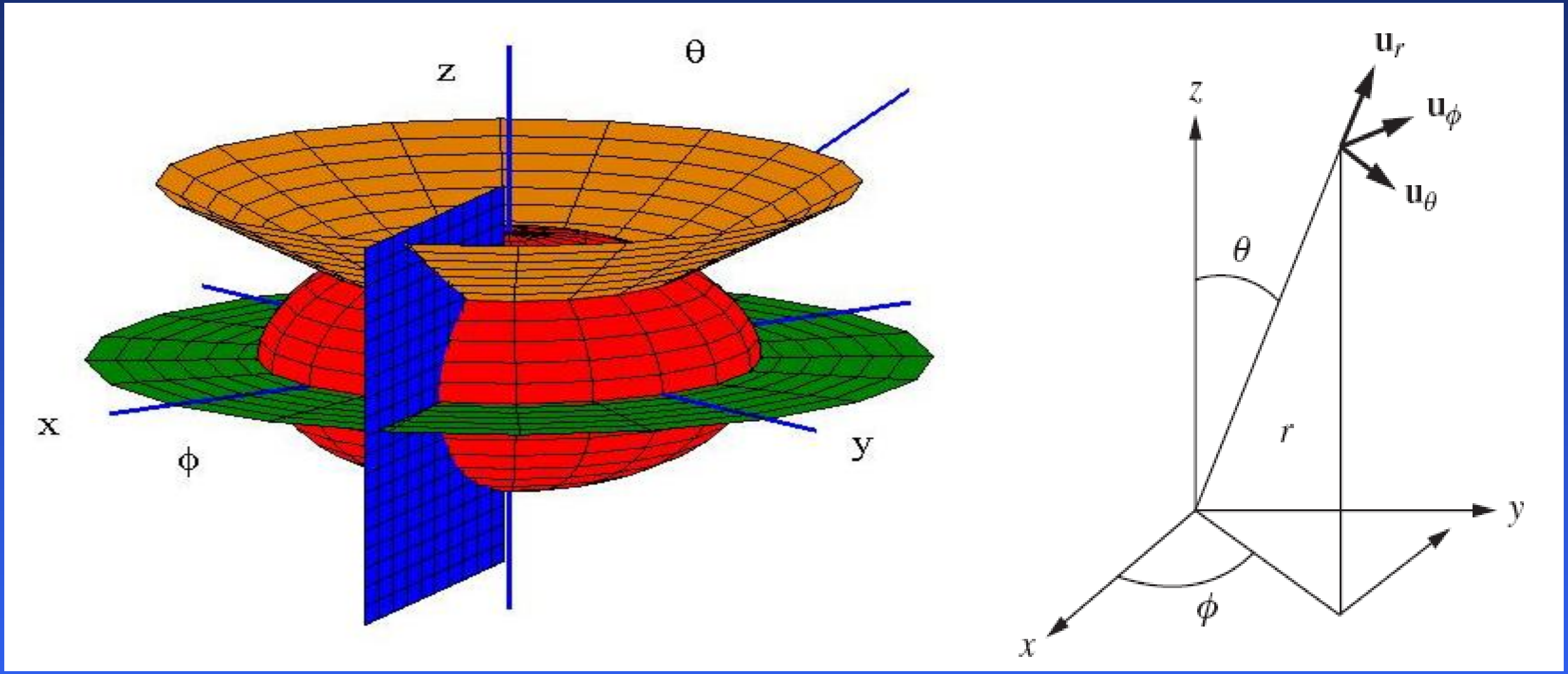
Diff. length: $dl = d\rho\, u_\rho + \rho\, d\phi\, u_\phi + dz\, u_z$ (2.12.1)

Diff. area: $ds = \rho\, d\phi\, dz\, u_\rho + d\rho\, dz\, u_\phi + \rho\, d\rho\, d\phi\, u_z$ (2.12.2)

Diff. volume: $dv = \rho\, d\rho\, d\phi\, dz$ (2.12.3)

An arbitrary vector: $A = A_\rho u_\rho + A_\phi u_\phi + A_z u_z$ (2.12.4)

Coordinate systems: Spherical



An intersection of a sphere of radius *r*, a plane that makes an angle ϕ to the *x* axis, and a cone that makes an angle θ to the *z* axis.

Coordinate systems: Spherical

Properties:
$$\left.\begin{aligned} u_r \times u_\theta &= u_\phi \\ u_\theta \times u_\phi &= u_r \\ u_\phi \times u_r &= u_\theta \end{aligned}\right\} \qquad (2.14.1)$$

Diff. length: $dl = dr\, u_r + r\, d\theta\, u_\theta + r \sin\theta\, d\phi\, u_\phi$ (2.14.2)

Diff. area: $ds = r^2 \sin\theta\, d\theta\, d\phi\, u_r + r \sin\theta\, dr\, d\phi\, u_\theta + r\, dr\, d\theta\, u_\phi$ (2.14.3)

Diff. volume: $dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$ (2.14.4)

An arbitrary vector: $A = A_r u_r + A_\theta u_\theta + A_\phi u_\phi$ (2.14.4)

System conversions

1. Cartesian to Cylindrical:

$$\rho = \sqrt{x^2 + y^2}; \quad \phi = \tan^{-1}\left(\frac{y}{x}\right); \quad z = z \tag{2.15.1}$$

2. Cartesian to Spherical:

$$r = \sqrt{x^2 + y^2 + z^2}; \quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right); \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{2.15.2}$$

3. Cylindrical to Cartesian:

$$x = \rho \cos\phi; \quad y = \rho \sin\phi; \quad z = z \tag{2.15.3}$$

4. Spherical to Cartesian:

$$x = r \sin\theta \cos\phi; \quad y = r \sin\theta \sin\phi; \quad z = r \cos\theta \tag{2.15.4}$$

cart2pol, cart2sph, pol2cart, sph2cart

Integral relations for vectors

1. Line integrals: $\int_a^b F \cdot dl \quad or \quad \oint F \cdot dl$

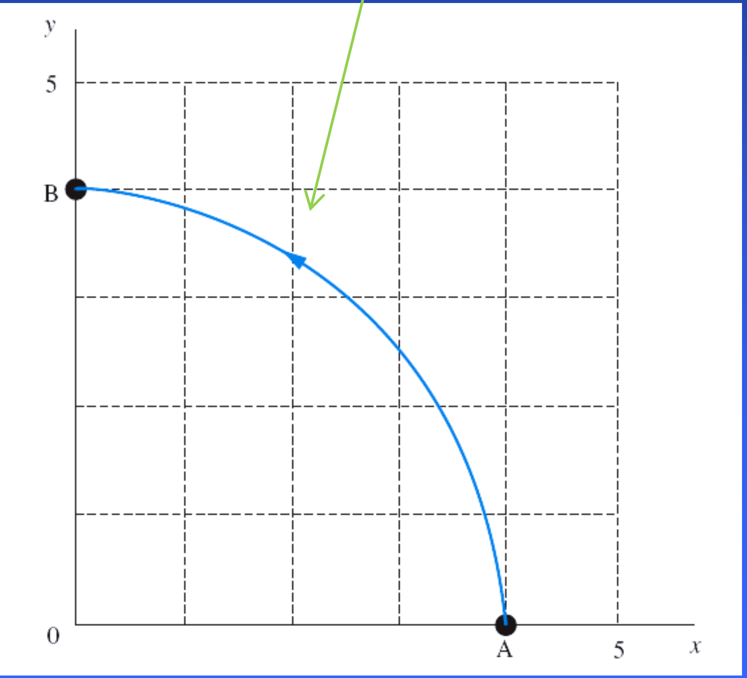
Example: calculate the work required to move a cart along the path from A to B if the force field is $F = 3xyu_x + 4xu_y$

$$\Delta W = \int_A^B F \cdot dl = \int_A^B (3xyu_x + 4xu_y) \cdot (dxu_x + dyu_y) =$$

$$\int_A^B (3xy\, dx + 4x\, dy)$$

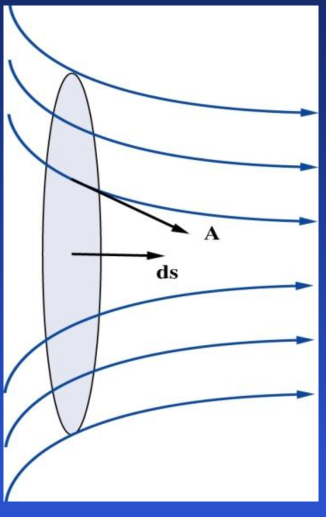
$$for\ the\ circle: x^2 + y^2 = 4^2$$

$$\Rightarrow \Delta W = \int_4^0 3x\sqrt{16 - x^2}\,dx + \int_0^4 4\sqrt{16 - y^2}\,dy = -64 + 16\pi$$



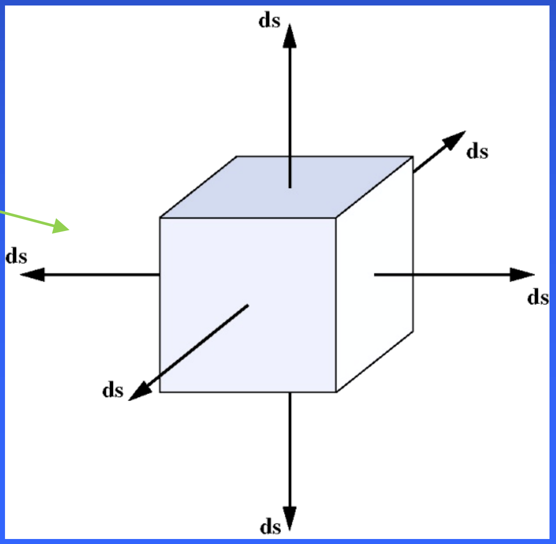
Integral relations for vectors (cont)

2. Surface integrals: $\int_{\Delta s} F \cdot ds \quad or \quad \oiint F \cdot ds$ *F* – a vector field



At the particular location of the loop, the component of **A** that is tangent to the loop does not pass through the loop. The scalar product **A • *ds*** eliminates its contribution.

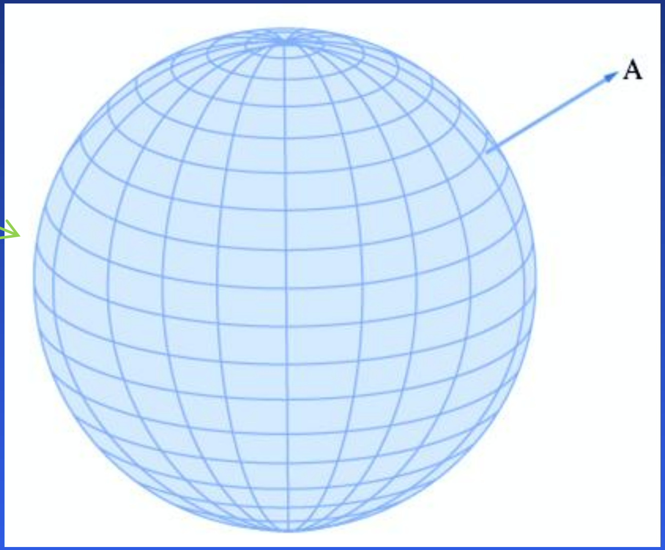
There are six differential surface vectors *ds* associated with the cube.



Here, the vectors in the *z*-plane:
$ds = dx\, dy\, u_z$ and $ds = dx\, dy\, (-u_z)$
are opposite to each other.

Integral relations for vectors (cont 2)

Example: Assuming that a vector field $A = A_0/r^2\, u_r$ exists in a region surrounding the origin, find the value of the closed-surface integral.



We need to use the differential surface area (in spherical coordinates) with the unit vector u_r since a vector field has a component in this direction only. From (2.14.4):

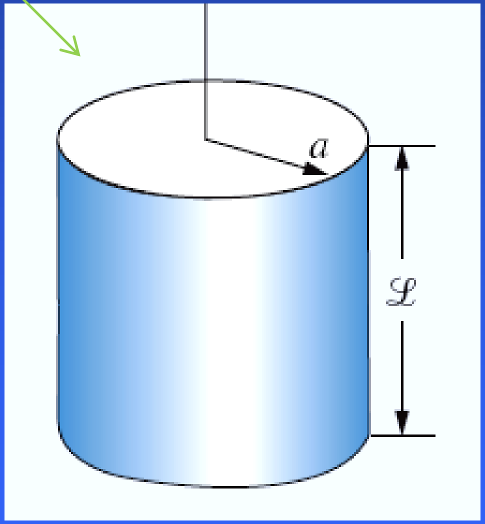
$$\oint A \cdot ds = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\frac{A_0}{r^2}u_r\right)\cdot\left(r^2\sin\theta\, d\theta\, d\phi\, u_r\right) = 4\pi A_0$$

Integral relations for vectors (cont 3)

3. Volume integrals: $\int\limits_{\Delta v} \rho_v dv$ ρ_v – a scalar quantity

Example: Find a volume of a cylinder of radius *a* and height *L*

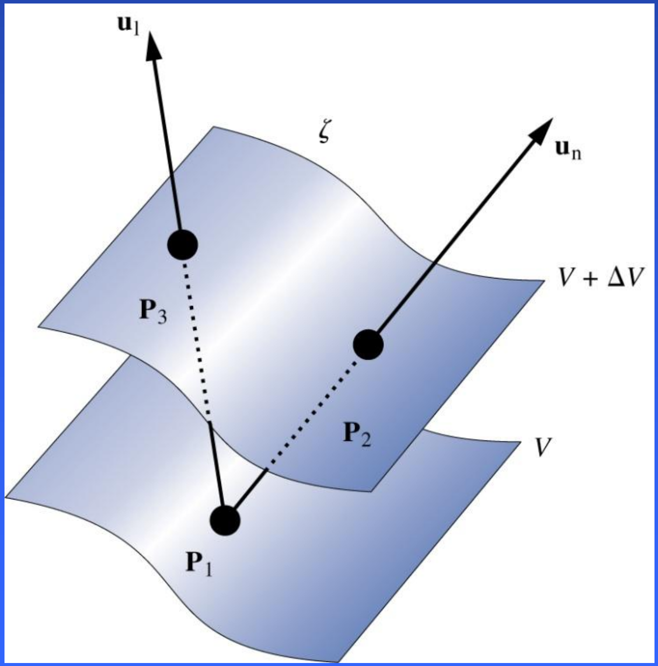
$$v = \int\limits_{\Delta v} dv = \int\limits_{z=0}^{L} \int\limits_{\phi=0}^{2\pi} \int\limits_{\rho=0}^{a} \rho \, d\rho \, d\phi \, dz = \pi a^2 L$$



Differential relations for vectors

1. Gradient of a scalar function: $$grad\left(a\right) \equiv \nabla a = \left(\frac{1}{h_1}\frac{\partial a}{\partial x_1}u_1, ..., \frac{1}{h_n}\frac{\partial a}{\partial x_n}u_n \right) \quad (2.20.1)$$

Gradient of a scalar field is a vector field which points in the direction of the greatest rate of increase of the scalar field, and whose magnitude is the greatest rate of change.



Two equipotential surfaces with potentials *V* and *V*+Δ*V*. Select 3 points such that distances between them $P_1P_2 \le P_1P_3$, i.e. $\Delta n \le \Delta l$.

$$\Rightarrow \frac{\Delta V}{\Delta n} \ge \frac{\Delta V}{\Delta l}$$

Assume that separation between surfaces is small:

$$\nabla V = \frac{\Delta V}{\Delta n}u_n \to \frac{dV}{dn}u_n$$

Projection of the gradient in the u_l direction:

$$\frac{\Delta V}{\Delta l}u_l \to \frac{dV}{dl}u_l = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\zeta = \frac{dV}{dn}u_n \cdot u_l = \nabla V \cdot u_l$$

Differential relations for vectors (cont)

Gradient in different coordinate systems:

$$1.\ Cartesian: \quad \nabla a = \left(\frac{\partial a}{\partial x} u_x, \frac{\partial a}{\partial y} u_y, \frac{\partial a}{\partial z} u_z \right) \tag{2.21.1}$$

$$2.\ Cylindrical: \quad \nabla a = \left(\frac{\partial a}{\partial \rho} u_\rho, \frac{1}{\rho}\frac{\partial a}{\partial \phi} u_\phi, \frac{\partial a}{\partial z} u_z \right) \tag{2.21.2}$$

$$3.\ Spherical: \quad \nabla a = \left(\frac{\partial a}{\partial r} u_r, \frac{1}{r}\frac{\partial a}{\partial \theta} u_\theta, \frac{1}{r \sin\theta}\frac{\partial a}{\partial \phi} u_\phi \right) \tag{2.21.3}$$

Example: $a(x, y, z) = 2x + 3y^2 - \sin(z);$

$$\Rightarrow \nabla a = \left(\frac{\partial a}{\partial x} u_x, \frac{\partial a}{\partial y} u_y, \frac{\partial a}{\partial z} u_z \right) = \left(2u_x, 6yu_y, -\cos(z)u_z \right)$$

gradient

Differential relations for vectors (cont 2)

2. Divergence of a vector field:

$$div\left(A\right) \equiv \nabla \cdot \mathrm{A} = \lim_{\Delta v \to 0} \frac{\oint A \cdot ds}{\Delta v} \qquad (2.22.1)$$

Divergence is an operator that measures the magnitude of a vector field's source or sink at a given point.

In different coordinate systems:

$$1.\, Cartesian: \quad \nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad (2.22.2)$$

$$2.\, Cylindrical: \quad \nabla \cdot A = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \qquad (2.22.3)$$

$$3.\, Spherical: \quad \nabla \cdot A = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \qquad (2.22.3)$$

divergence

Differential relations for vectors (cont 3)

Example: $A(x,y,z) = 2x\,u_x + 3y^2 u_y - \sin(z)\,u_z;$

$$\Rightarrow \nabla\cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2 + 6y - \cos(z)$$

Some "divergence rules":

$$\nabla\cdot a = 0; \qquad (2.23.1)$$

$$\nabla\cdot(A_1 + A_2) = \nabla\cdot A_1 + \nabla\cdot A_2; \qquad (2.23.2)$$

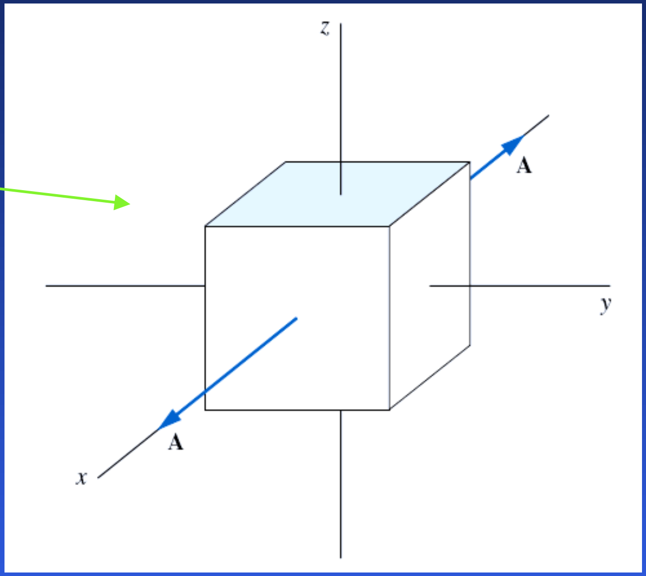
$$\nabla\cdot cA = c\nabla\cdot A \qquad (2.23.3)$$

Divergence (Gauss's) theorem:

$$\oiint A\cdot ds = \int_{v} \left(\nabla\cdot A\right) dv \qquad (2.23.4)$$

Differential relations for vectors (cont 4)

Example: evaluate both sides of Gauss's theorem for a vector field: $A = x\ u_x$ within the unit cube



$$\Rightarrow \quad \frac{\partial A_x}{\partial x} = 1; \quad \frac{\partial A_y}{\partial y} = 0; \quad \frac{\partial A_z}{\partial z} = 0.$$

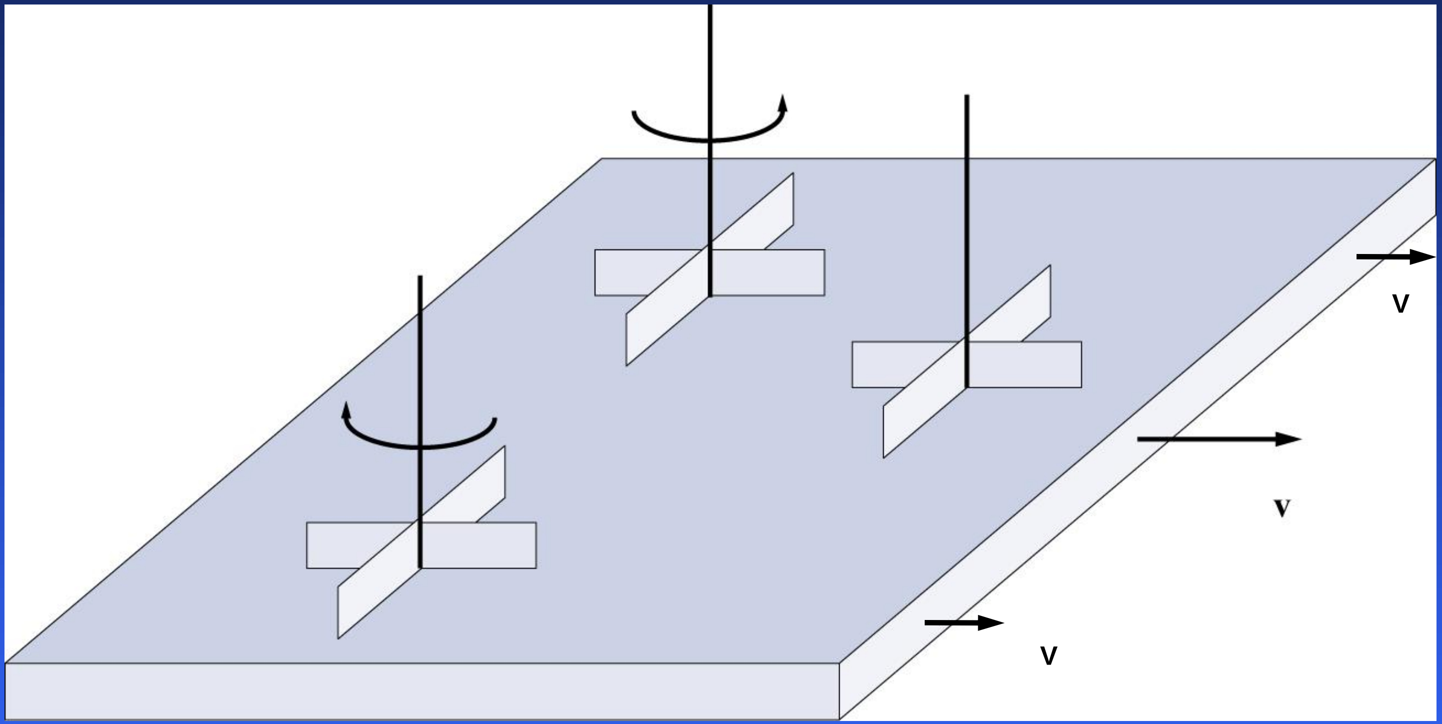
The volume integral is:

$$\int_{\Delta v} \nabla \cdot A\, dv = \int_{x=-1/2}^{1/2} \int_{y=-1/2}^{1/2} \int_{z=-1/2}^{1/2} dx\, dy\, dz = 1$$

The closed-surface integral is:

$$\oiint A \cdot ds = \int_{y=-1/2}^{1/2} \int_{z=-1/2}^{1/2} \left(x\,|_{x=-1/2} \right) u_x \cdot (-dz\, dy\, u_x) + \int_{y=-1/2}^{1/2} \int_{z=-1/2}^{1/2} \left(x\,|_{x=1/2} \right) u_x \cdot (dz\, dy\, u_x) = 1$$

Differential relations for vectors (cont 5)



Assume we insert small paddle wheels in a flowing river.
The flow is higher close to the center and slower at the edges.
Therefore, a wheel close to the center (of a river) will not rotate since velocity of water is the same on both sides of the wheel.
Wheels close to the edges will rotate due to difference in velocities.
The curl operation determines the direction and the magnitude of rotation.

Differential relations for vectors (cont 6)

3. Curl of a vector field: $$curl\,(A) \equiv \nabla \times A = \lim_{\Delta s \to 0} \frac{u_n \oint A \cdot dl}{\Delta s} \qquad (2.26.1)$$

Curl is a vector field with magnitude equal to the maximum "circulation" at each point and oriented perpendicularly to this plane of circulation for each point. More precisely, the magnitude of curl is the limiting value of circulation per unit area.

In different coordinate systems:

$$1.\,Cartesian: \quad \nabla \times A = \begin{vmatrix} u_x & u_y & u_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) u_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) u_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) u_z \qquad (2.26.2)$$

$$2.\,Cylindrical: \nabla \times A = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) u_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) u_\phi + \left(\frac{1}{\rho}\frac{\partial\left(\rho A_\phi\right)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right) u_z \qquad (2.26.3)$$

$$3.\,Spherical: \quad \nabla \times A = \left[\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\right] u_r + \left[\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(r\,A_\phi)}{\partial r}\right)\right] u_\theta + \left[\frac{1}{r}\left(\frac{\partial(r\,A_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\right] u_\phi$$

curl

(2.26.3)

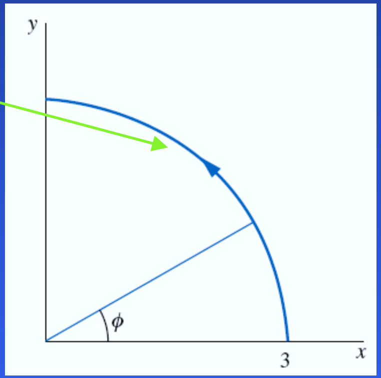
Differential relations for vectors (cont 7)

Stokes' theorem:

$$\oint A\cdot dl = \int_{\Delta S} \nabla\times A\cdot ds \qquad (2.27.1)$$

The surface integral of the curl of a vector field over a surface ΔS equals to the line integral of the vector field over its boundary.

Example: For a v. field $A = -xy\, u_x - 2x\, u_y$, verify Stokes' thm. over



$$from\ (2.26.2) \Rightarrow \nabla\times A = -(2+x)u_z$$

$$\int_{\Delta S} \nabla\times A\cdot ds = \int_{y=0}^{3}\int_{x=0}^{\sqrt{9-y^2}} \nabla\times A\cdot dx\,dy\,u_z = \int_{y=0}^{3}\int_{x=0}^{\sqrt{9-y^2}} -(2+x)u_z\cdot dx\,dy\,u_z$$

$$= \int_{y=0}^{3}\left(2\sqrt{9-y^2}+\frac{9-y^2}{2}\right)dy = -9\left(1+\frac{\pi}{2}\right)$$

$$\oint A\cdot dl = \cancel{\int_{x=0}^{3}\int_{y=0}^{0} A\cdot dx\,u_x} + \int_{arc} A\cdot dl + \cancel{\int_{x=0}^{0}\int_{y=3}^{0} A\cdot dy\,u_y} = \int_{arc}\left(xy\,u_x - 2x\,u_y\right)\cdot\left(dx\,u_x + dy\,u_y + dz\,u_z\right)$$

$$= \int_{arc}(xy\,dx - 2x\,dy) = \int_{3}^{0} x\sqrt{9-x^2}\,dx - 2\int_{0}^{3}\sqrt{9-y^2}\,dy = -9\left(1+\frac{\pi}{2}\right)$$

Repeated vector operations

$$\nabla \cdot \nabla \times A = 0 \quad (2.28.1)$$

$$\nabla \times \nabla a = 0 \quad (2.28.2)$$

$$\nabla \cdot \nabla a = \nabla^2 a \quad (2.28.3)$$

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A \quad (2.28.4)$$

The Laplacian operator:

$$\nabla^2 a = \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2} \qquad \text{Cartesian} \quad (2.28.5)$$

$$= \frac{1}{\rho}\frac{\partial\left(\rho \frac{\partial a}{\partial \rho}\right)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 a}{\partial \phi^2} + \frac{\partial^2 a}{\partial z^2} \qquad \text{Cylindrical} \quad (2.28.6)$$

$$= \frac{1}{r^2}\frac{\partial\left(r^2 \frac{\partial a}{\partial r}\right)}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial\left(\sin\theta \frac{\partial a}{\partial \theta}\right)}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 a}{\partial \phi^2} \qquad \text{Spherical} \quad (2.28.7)$$

Phasors

A **phasor** is a constant complex number representing the complex amplitude (magnitude and phase) of a sinusoidal function of time.

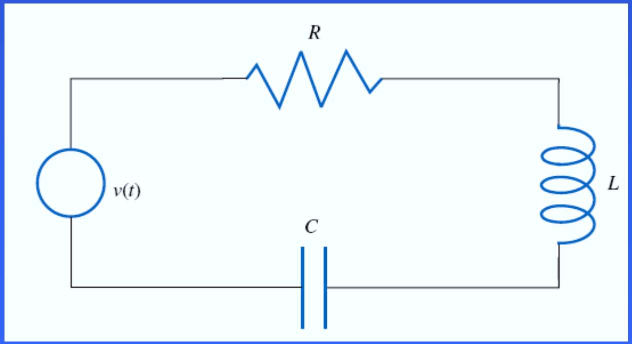
$$V(\chi) = V_0 e^{j\varphi} \quad (2.29.1)$$

$$for \quad v(t) = V_0 \cos(\omega t + \varphi) = \mathrm{Re}\left[V(\chi) e^{j\omega t}\right] \quad (2.29.2)$$

$$\Rightarrow \quad \frac{dv}{dt} = \mathrm{Re}\left[j\omega V(\chi) e^{j\omega t}\right]; \quad \int v\, dt' = \mathrm{Re}\left[\frac{1}{j\omega} V(\chi) e^{j\omega t}\right] \quad (2.29.3)$$

Note: Phasor notation implies that signals have the same frequency. Therefore, phasors are used for linear systems…

Example: Express the loop eqn for a circuit in phasors if $v(t) = V_0 \cos(\omega t)$



$$v = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\, dt' \qquad i(t) = I_0 \cos(\omega t + \varphi)$$

$$\Rightarrow V_0 \cos(\omega t) = I_0\left[-\omega L \sin(\omega t + \varphi) + R\cos(\omega t + \varphi) + \frac{1}{\omega C}\sin(\omega t + \varphi)\right]$$

$$v(t) = V_0 \cos(\omega t) = \mathrm{Re}\left[V_0 e^{j\varphi} e^{j\omega t}\right] = \mathrm{Re}\left[V(\chi) e^{j\omega t}\right]$$

$$V(\chi) = \left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right] I(\chi)$$

Conclusions

Questions?

Ready for your first homework??