

Calabi - Yau manifolds

What is CY?

- complex : one can define holomorphic coordinates x^i ($i=1, \dots, n$)

- Kähler : $g_{ij} = 0$, $g_{i\bar{j}} = 0$

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$$

- Ricci flat : $R_{i\bar{j}} = 0$

For a Kähler manifold, $R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log \det g$.

$$\Rightarrow R_{i\bar{j}} = 0 \text{ means } \det g = \Omega \bar{\Omega}$$

Ω : holomorphic on M .

transforms as (n,0)-form

$M : \text{CY} \Rightarrow \exists$ no-where vanishing (n,0)-form
($\Leftrightarrow C_1 \neq 0$)



conjectured by Calabi

proven by Yau

example of CY

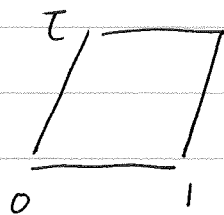
1d (complex 1d)

• \mathbb{C}

• $\mathbb{C}^\times = \mathbb{C} \setminus \{0\} \sim \text{cylinder}$

• $0 \text{ --- } \infty$

• $\mathbb{T}^2 = \mathbb{C} / \{ \mathbb{Z} + \tau \mathbb{Z} \}$



$$x = x' + \tau x''$$

$$0 \leq x', x'' \leq 1$$

τ : complex structure

$$dx = dx' + \tau dx'' ; (1,0) \text{ form}$$

$$1 = \int_0^1 dx'$$



$$\tau = \int_0^\tau dx''$$



period
integral.

$$(h^{p,q}) \quad \begin{matrix} & h^{0,0} & & \\ h^{1,0} & & h^{0,1} & \\ & h^{1,1} & & \end{matrix} = \begin{matrix} & & & 1 \\ & & 1 & \\ & & & 1 \end{matrix}$$

r : area of T^2 .

$$\begin{aligned} \text{metric } g_{ij} dx^i dx^j &= r dx d\bar{x} / \text{Im } \tau \\ &= r (dx^1 dx^1 + 2\text{Re } \tau dx^1 dx^2 \\ &\quad + |\tau|^2 dx^2 dx^2) / \text{Im } \tau. \end{aligned}$$

Kähler form

$$k = \frac{i}{2} g_{i\bar{j}} dx^i d\bar{x}^j = r dx^1 \wedge dx^2.$$

r : Kähler class

Yau's theorem

M : $C_1 = 0$, given complex str
and Kähler class

\Rightarrow

\exists unique CY metric.

String theory r : complexify $t = i'r + \theta$
Kähler moduli

$$(\tau, t) \in \frac{O(2,2)}{O(2,2;\mathbb{Z})} / O(2) \times O(2)$$

2 complex dim

There are only 2 classes of compact CY's

• T^4

$$\begin{array}{ccccc}
 & & h^{00} & & \\
 & & h^{10} & h^{01} & \\
 h^{20} & h^{11} & h^{02} & & \\
 & h^{21} & h^{12} & & \\
 & & h^{22} & &
 \end{array}
 =
 \begin{array}{ccccc}
 & & & & 1 \\
 & & 2 & & 2 \\
 & 1 & & 4 & 1 \\
 & & 2 & & 2 \\
 & & & & 1
 \end{array}$$

complex moduli space $4\mathbb{C}$.

Kähler class $4 \in h^{1,1} = 4$.

\swarrow
 strings $4\mathbb{C}$ $8\mathbb{C}$.
 theory

They combine to make $\backslash O(4,4) / O(4) \times O(4)$
 $O(4,4;\mathbb{Z})$.

In general, the moduli space for string theory
 on T^D is

$$O(D,D;\mathbb{Z}) \backslash O(D,D) / O(D) \times O(D)$$

K_3 : only non-flat compact CI_2

$$(h^{p,q}) = \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

$$\chi = 24$$

$$h^{1,0} = 0 \Rightarrow \text{no isometry}$$

moduli space of metrics = locally $O(19,3)/O(19) \times O(3)$

A Todorov.

~~Free~~ Complex moduli $\Leftrightarrow h^{1,1} = 20$ complex dim

stringy Kähler moduli $\Leftrightarrow h^{1,1} = 20$.

$$O(20,4; \mathbb{Z}) \setminus O(20,4) / O(20) \times O(4)$$

40 complex dim

examples

start with \mathbb{CP}^3

$$\text{i.e. } (z_1 \dots z_4) \sim \lambda (z_1 \dots z_4), \lambda \in \mathbb{C}^\times$$

Consider homogeneous function $P(z)$
of degree d .

$$P(z_1 \dots z_4) = 0.$$

In general, a hypersurface X in \mathbb{CP}^{k-1}

$$\text{defined by } \underset{\substack{\uparrow \\ \text{deg } d}}{P(z_1 \dots z_k)} = 0$$

$$\text{has } C_1 \sim (d-k) C_1(\mathbb{CP}^{k-1})$$

So, we need $d=k$ for CY.

in our case $d=4$

$$\text{e.g. } (X_1)^4 + (X_2)^4 + (X_3)^4 + (X_4)^4 = 0$$

in \mathbb{CP}^4

• CY_3

$$h^{p,q} = \begin{array}{ccccccc} & & & & 1 & & \\ & & & 0 & & 0 & \\ & & 0 & & h^{1,1} & & 0 \\ & 1 & h^{1,2} & & h^{2,1} & & 1 \\ & & 0 & h^{2,2} & & 0 & \\ & & 0 & 0 & & & \\ & & & & 1 & & \end{array}$$

$$h^{1,1} = h^{2,2} \quad , \quad h^{2,1} = h^{1,2} \quad \text{duality.}$$

$$h^{1,0} = 0 \quad \text{no isometry}$$

$$(h^{2,0} = h^{0,2} = h^{0,1} = 0 \text{ follows})$$

by complex conjugate + duality.)

$$\text{If } h^{1,0} \neq 0 \Rightarrow \text{torus.}$$

$$\chi = 2(h^{1,1} - h^{2,1})$$

$$\text{complex structure deformation} \Leftrightarrow h^{2,1}$$

$$\text{Kähler deformation} \Leftrightarrow h^{1,1}$$

$$(\text{In general complex deformation} \Leftrightarrow h^{d-1,1})$$

Complex structure moduli space M_C

holomorphic $(3,0)$ -form Ω

Ω defines a line bundle over M_C
(sub-bundle of the Hodge bundle)

with a metric $\|\Omega\|^2 = i \int_M \Omega \wedge \bar{\Omega}$

Define $K = -\log \|\Omega\|^2$

Then M_C becomes a Kähler mfd.

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K.$$

• flat coordinates on M_C .

Choose a basis $\{\alpha_I, \beta^I\}_{I=0,1,\dots,h^{2,1}}$

of $H_3(M, \mathbb{Z})$

$$h_3 = 2 + 2h^{2,1}$$

Define the periods:

$$X^I = \int_{\alpha_I} \Omega, \quad F_I = \int_{\beta^I} \Omega$$

9.

$F_I = F_I(x)$ homogeneous, degree 1

$$\|\Omega\|^2 = X^I \bar{F}_I - \bar{X}^{\bar{I}} F_{\bar{I}}$$

$$\exists F, \quad F_I = \partial_I F(x)$$

↑ pre-potential (\Rightarrow Seiberg-Witten theory)

$$t^a = \frac{X^a}{X^0} \quad a=1, \dots, h^{2,1}$$

One can show

$$K = -\log(4F - 4\bar{F} + \bar{t}^a \partial_a F - t^a \partial_a \bar{F})$$

$$R_{a\bar{b}c\bar{d}} = G_{a\bar{b}} G_{c\bar{d}} + G_{a\bar{d}} G_{c\bar{b}}$$

$$- e^{2K} C_{ace} \bar{C}_{\bar{b}\bar{d}\bar{f}} G^{e\bar{f}}$$

$$\text{where } C_{abc} = \frac{\partial^3 F}{\partial t^a \partial t^b \partial t^c}$$

\Downarrow

$$[V-C, V-C] = 0$$

non-compact examples

1. local \mathbb{CP}^2

\mathbb{CP}^2 is not CY, Consider a line bundle over \mathbb{CP}^1 .

so that the 1st Chern class of the fiber cancels that of the base.

\Rightarrow start with $(x, z_1, z_2, z_3) \in \mathbb{C}^4 \setminus \{0\}$

$$(x, z_1, z_2, z_3) \sim (\lambda^{-3} x, \lambda z_1, \lambda z_2, \lambda z_3)$$

Weighted projective space

It is a total space of $\mathcal{O}(-3) \rightarrow \mathbb{CP}^1$.

2. local \mathbb{CP}^1

$$(x_1, x_2, z_1, z_2) \sim (\lambda^{-1} x_1, \lambda^{-1} x_2, \lambda z_1, \lambda z_2)$$

$$\Rightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{CP}^1$$

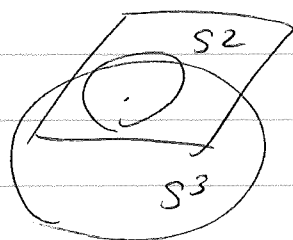
3. conifold

$$(x, y, w, z) \in \mathbb{C}^4$$

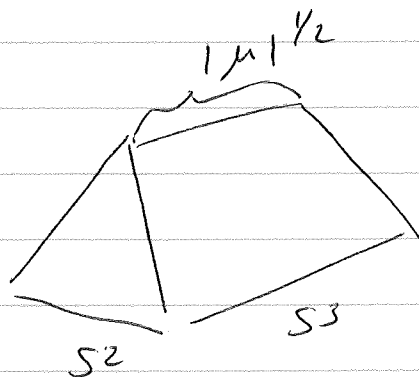
$$xy - wz = \mu \quad \mu : \text{complex modulus}$$

This is the same as T^*S^3

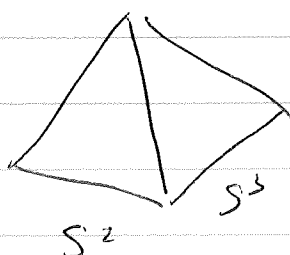
$$(\text{radius } S^3 \sim \sqrt{|\mu|})$$



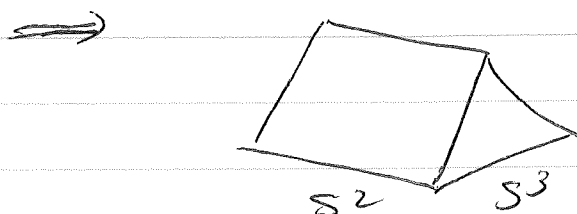
\Rightarrow the infinity of T^*S^3
looks like $S^2 \times S^3$



$\mu \rightarrow 0$: conifold singularity



resolution



toric CYs

example local \mathbb{CP}^2

$$(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \setminus \{0\}$$

$$(Q_0, Q_1, Q_2, Q_3) \equiv (-3, 1, 1, 1) \text{ "charge"}$$

$$\text{Consider } z_i \rightarrow e^{iQ_i \theta} z_i \quad i=0,1,2,3$$

If \mathbb{C}^4 is given with the Kähler form

$$K = i \sum_{i=0}^3 dz_i \wedge d\bar{z}_i$$

the $U(1)$ symmetry is generated by

$$CP \equiv -3|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2$$

$$\text{local } \mathbb{CP}^2 = \varphi^{-1}(r) / U(1)$$

Define $z_i = \sqrt{p_i} e^{i\phi_i}$ $p_i \geq 0$

$$k = i \sum_i dz_i \wedge d\bar{z}_i = i \sum_i dp_i \wedge d\phi_i$$

$$\varphi = -3p_0 + p_1 + p_2 + p_3 = r.$$

$$(\phi_0, \phi_1, \phi_2, \phi_3) \sim (\phi_0 - 3\theta, \phi_1 + \theta, \phi_2 + \theta, \phi_3 + \theta)$$

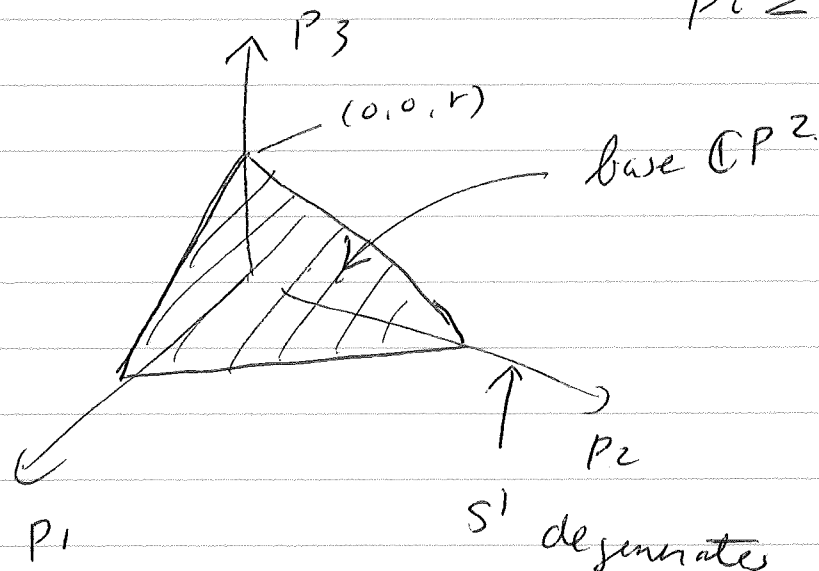
We can eliminate (p_0, ϕ_0)

$\Rightarrow T^3(\phi_1, \phi_2, \phi_3)$ fibered over

$$(p_1, p_2, p_3)$$

$$p_1 + p_2 + p_3 \geq r$$

$$p_i \geq 0$$



In general,

- Start with $(z_1, \dots, z_{N+3}) \in \mathbb{C}^{N+3} \setminus \{0\}$
- Divide by $U(1)^N$

$$Q^1 = (Q_1^1 \dots Q_{N+3}^1)$$

;

$$Q^N = (Q_1^N \dots Q_{N+3}^N)$$

- ~~the~~ $\varphi_a = \sum_{i=1}^{N+3} Q_i^a |z_i|^2$

$$\varphi_a = t^a : \text{Kähler models}$$

$$\text{toric } CY_3 = \varphi^{-1}(t) / U(1)^N$$

local \mathbb{CP}^1 .

$$Q = (-1, -1, 1, 1)$$

Question: Draw its triic diagram