Discrete Mathematics and Its Applications

Introduction to Logic

Maryam Al-Towailb

King Saud University

Math. 151 - Math. 1101

Outline

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Propositional Logic

Definition

A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Example

1. Washington, D.C., is the capital of the United States of America
2. $3 + 2 = 6$
3. What time is it?
4. Read this carefully.
5. $x + 1 = 2$

Propositions 1 is true, whereas 2 is false. Sentences 3 and 4 are not propositions because they are not declarative sentences. Sentence 5 is not propositions because they are neither true nor false.

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Propositional Logic

The truth value of a proposition is true, denoted by T, if it is a true proposition and false, denoted by F, if it is a false proposition.

Definition

Let p be a proposition. The negation of p, denoted by $\neg p$ (also denoted by $\bar{p}$), is the statement "It is not the case that p." The proposition $\neg p$ is read "not p." The truth value of the negation of p, is the opposite of the truth value of p.

Example

Find the negation of the proposition "Today is Friday" and express this in simple English. The negation is "It is not the case that today is Friday". This negation can be more simply expressed by "Today is not Friday," or "It is not Friday today."

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The truth table for the negation of a proposition p.

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TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

Propositional Logic

Definition

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Example

Find the conjunction of the propositions p and q where p is the proposition "Today is Friday" and q is the proposition "It is raining today." The conjunction of these propositions, $p \wedge q$, is the proposition "Today is Friday and it is raining today."

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Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Example

What is the disjunction of the propositions p and q where p is the proposition "Today is Friday" and q is the proposition "It is raining today." The disjunction of p and q, $p \vee q$, is the proposition "Today is Friday or it is raining today."

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The truth table for the negation of a proposition p.

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TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition

Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
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F	T	T
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Let p and q be propositions. The conditional statement $p \rightarrow q$ is the proposition "if p, then q." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence)

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Let p and q be propositions. The conditional statement $p \to q$ is the proposition "if p, then q ." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence)

TABLE 5 The Truth Table for the Conditional Statement $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

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Truth Tables o f Compound Propositions.

Converse, Contrapositive and inverse

1. The proposition $q \rightarrow p$ is called the converse of $p \rightarrow q$.
2. The contrapositive of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.
3. The proposition $\neg p \rightarrow \neg q$ is called the inverse of $p \rightarrow q$.

We will see that of these three conditional statements formed from $p \rightarrow q$, only the contrapositive always has the same truth value as $p \rightarrow q$.
When two compound propositions always have the same truth value we call them equivalent,

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Definition

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
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Truth Tables o f Compound Propositions:

Example

Construct the truth table of the following compound propositions:

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

TABLE 7 The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

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Propositional Equivalences:

Definition

A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a tautology.
A compound proposition that is always false is called a contradiction.
A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Example

We can construct examples of tautologies and contradictions using just one propositional variable.
Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$. Because $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction

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Propositional Logic

Logical Equivalences:

Definition

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

In particular, the compound propositions p and q are equivalent if and only if the columns giving their truth values agree.

TABLE 2 De Morgan's Laws.
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

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Example

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

TABLE 3 Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

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TABLE 3 **Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.**

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
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Propositional Logic

Example

Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.

TABLE 4 Truth Tables for $\neg p \vee q$ and $p \to q$.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
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9.jpg

Propositional Logic

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

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