PHYS 404

Lecture 1: Legendre Functions

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Legendre Functions – physical justification

- Legendre functions or Legendre polynomials are the solutions of Legendre's differential equation that appear when we separate the variables of Helmholtz' equation, Laplace equation or Schrodinger equation using spherical coordinates.

Legendre Functions – physical justification

- Almost all the cases are specific cases of the general differential equation:

$$\left\{\nabla^2 + k^2 - U(r)\right\}\Psi\left(\mathbf{r}\right) = 0, \quad k = const \qquad (1)$$

- Following the method of separation of variables, i.e. by considering a solution of the form:

$$\Psi\left(\mathbf{r}\right) = R\left(r\right)Y\left(\theta,\varphi\right) \qquad (2)$$

Legendre Functions – physical justification

- The differential equation (1) splits in two simpler equations

$$\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}Y\left(\theta,\varphi\right)=-\lambda Y\left(\theta,\varphi\right)\quad(1.3)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[k^2-U(r)-\frac{\lambda}{r^2}\right]R=0\quad(1.4)$$

Legendre Functions – physical justification

- Eq. 3 is independent from function $U(r)$ and is called **angular** equation. Solutions of this equation could be: trigonometric functions, Legendre functions, associated Legendre functions and the spherical harmonics.
- Eq. 4 does depend on function $U(r)$ and is called **radial** equation. Solutions of this equation could be: Bessel functions, Hermite functions and Laguerre functions.

The angular equation

- Try to consider for Eq. 3 a solution of the separating variable form:

$$Y(\theta,\varphi)=\Theta(\theta)\Phi(\varphi) \qquad (1.5)$$

- Then the radial equation is split into two equations:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(\lambda-\frac{\mu^2}{\sin^2\theta}\right)\Theta=0 \qquad (1.6)$$

$$\frac{d^2\Phi}{d^2\phi}+\mu^2\Phi=0 \qquad (1.7)$$

The associated Legendre equation

- If we make the substitutions

$$x = \cos\theta, \quad u(x) = \Theta\left(\theta\right), \quad \lambda = \nu(\nu+1) \qquad (1.8)$$

Then Eq. 1.7 takes the form

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{du}{dx}\right] + \left[\nu(\nu+1) - \frac{\mu^2}{1-x^2}\right]u = 0 \quad (1.9)$$

This is the so called **associated Legendre equation**

Substitution x=cosθ is needed in order to give us a DE with rational coefficients for which we have "efficient tools" for its solution; method of power series

The associated Legendre equation

- The associated Legendre Equation has two linearly independent solutions:
- The associated Legendre function of first kind: $P_{\mu}^{\nu}\left(x\right)$
- The associated Legendre function of second kind: $Q_{\mu}^{\nu}\left(x\right)$
- In physical problems Eq. 1.7 has single valued and periodical solutions only if *m =integer*. In this case

$$\Phi_m = \exp\left(im\phi\right) / \sqrt{2\pi} \qquad (1.10)$$

The associated Legendre equation

- Then in this case the associated Legendre Equation is written as

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{du}{dx}\right]+\left[\nu(\nu+1)-\frac{m^2}{1-x^2}\right]u=0 \quad (1.10)$$

Legendre Diff. Equation

- If in the previous equation we consider m=0 and also consider that $u(x)\big|_{m=0} = y(x)$ then we get the **Legendre differential equation**:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]+\nu(\nu+1)y=0 \quad (1.11)$$

- Eq. 1.11 has finite solutions at the points where $x=\pm 1$ only if $\nu = n$ is a positive integer.

Legendre Functions

- Legendre functions (or polynomials) $P_n(x)$, $Q_n(x)$ are a solution of Legendre differential equation about the origin ($x = 0$).
- The polynomials $Q_\ell(x)$ are rarely used in physics problems so we are not going to deal with them further.
- On the contrary the polynomials $P_\ell(x)$, which appear in many physical problems, may be defined by the so called **generating function**.

Legendre Polynomial
The generating function

The *generating* function of Legendre polynomials:

$$g(t,x) = (1 - 2xt + t^2)^{-1/2} \qquad (1.12)$$

Which has an important application in **electric multipole expansions**. If we expand this function as a binomial series if $|t| < 1$ we obtain

$$g(t,x) = \sum_{n=0}^{\infty} P_n(x)\, t^n \qquad |t| < 1 \qquad (1.13)$$

Legendre Polynomial

The generating function

Using the binomial theorem we can expand the generating function as follows:

$$g(t,x) = \left(1 - 2xt + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} \frac{\left(2n\right)!}{2^{2n}\left(n!\right)^2}\left(2xt - t^2\right)^n \qquad (1.14)$$

From the above series we can get the values of the Legendre functions.

Legendre Polynomial

plots

$$P_0(x) = 1$$

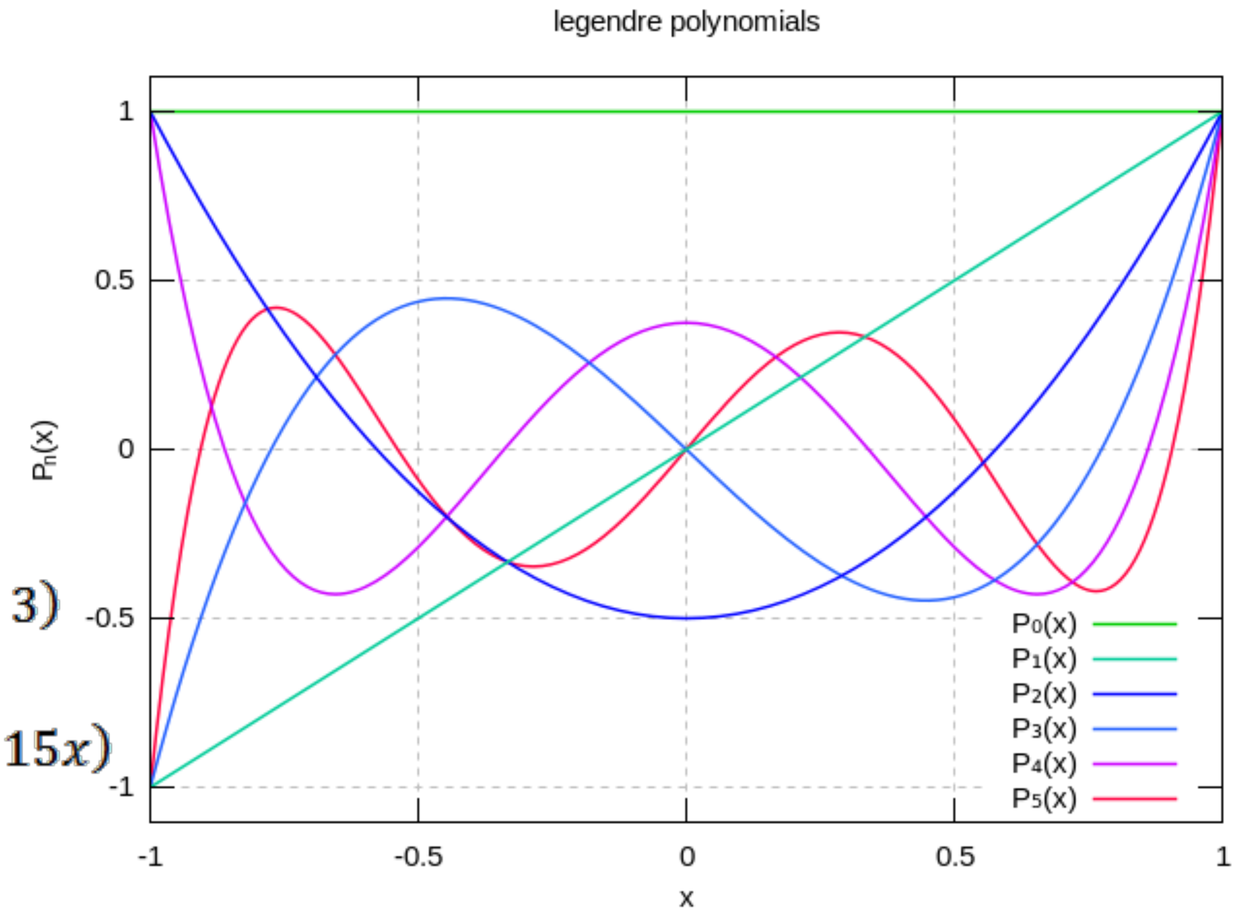
$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$



The Recurrence Relations

Legendre polynomials obey the following recurrence relations:

$$\left(2n+1\right)xP_n(x) = \left(n+1\right)P_{n+1}(x) + nP_{n-1}(x)$$

$$P_{n+1}^{'}(x) + P_{n-1}^{'}(x) = 2xP_n^{'}(x) + P_n(x)$$

$$P_{n+1}^{'}(x) - P_{n-1}^{'}(x) = \left(2n+1\right)P_n(x)$$

$$P_{n+1}^{'}(x) = \left(n+1\right)P_n(x) + xP_n^{'}(x)$$

$$P_{n-1}^{'}(x) = -nP_n(x) + xP_n^{'}(x)$$

$$\left(1-x^2\right)P_n^{'}(x) = nP_{n-1}(x) - nxP_n(x) \qquad (1.15)$$

These relations are valid for $n = 1,2,3,\ldots$

Legendre Polynomials

Special Properties

- Some special values
- $$P_n(1) = 1 \qquad (1.6a)$$

$$P_n(-1) = (-1)^n \qquad (1.16b)$$

and

$$\left.\begin{aligned} P_{2n}(0) &= (-1)^n \frac{(2n-1)!!}{(2n)!!} \\ P_{2n+1}(0) &= 0 \end{aligned}\right\} \quad \text{for } n = 0,1,2,\ldots$$

Where $n!! = \begin{cases} n(n-2)(n-4)\ldots 1 \text{ if } n \, is \, odd \\ n(n-2)(n-4)\ldots 2 \text{ if } n \, is \, even \\ 1 \qquad\qquad\qquad\quad \text{if } n = 0 \end{cases}$ (called double factorial)

Legendre Polynomials
Special Properties

- The Parity property:(with respect to $x = 0 \ , \theta = \pi/2$)

$$P_n(-x) = (-1)^n P_n(x) \qquad (1.17a)$$

$$P_n(\cos(\pi - \theta)) = (-1)^n P_n(\cos(\theta)) \qquad (1.17b)$$

If n is odd the parity of the polynomial is odd, but if it is even the parity of the polynomial is even.

- Upper and lower Bounds for $P_n(\cos(\theta))$

$$|P_n(\cos(\theta))| \leq P_n(1) = 1 \qquad (1.18)$$

Legendre Polynomials

Orthogonality

Legendre's equation is a self-adjoint equation, which satisfies Sturm-Liouiville theory, where the solutions are expected to be orthogonal to satisfying certain boundary conditions. Legendre polynomials are a set of orthogonal functions on (-1,1).

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{n,m} \qquad (1.19)$$

where

$$\delta_{n,m} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

Legendre Polynomials

Legendre Series

- According to Sturm-Liouville theory that Legendre polynomial form a complete set.
- $$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \qquad (1.20)$$
- The coefficients a_n are obtained by multiplying the series by $P_m(x)$ and integrating in the interval [-1,1]

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x) \qquad (1.21)$$

Alternate definitions of Legendre polynomials

- Legendre polynomials can be defined with the help of the so called **Rodriguez formula**:

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n \left(x^2 - 1\right)^n \qquad (1.22)$$

- The Rodrigues' formula provides a means of developing a means of developing an integral representation in the complex plane with the **Schlaefli** integral :

$$P_n(z) = \frac{1}{2^n n!}\left(\frac{d}{dz}\right)^n \left(z^2 - 1\right)^n = \frac{2^{-n}}{2\pi i}\oint \frac{\left(t^2 - 1\right)^n}{\left(t - z\right)^{n+1}} dt$$

Contour encloses *t*=*z*.