PHYS 404

Lecture 2: Associated Legendre Functions

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Associated Legendre Functions

- As we have seen the associated Legendre d.e. is given by Eq. (1.9):

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{du}{dx}\right]+\left[n(n+1)-\frac{m^2}{1-x^2}\right]u=0 \quad (2.1)$$

- We can prove that the solution of (2.1) is written as:

$$u(x)=\left(1-x^2\right)^{m/2}\frac{d^m y(x)}{dx^m}, \quad (2.2)$$

- Where *y(x)* is a solution of Legendre d.e. (1.11)

Associated Legendre Functions

- The d.e. has two linearly independent solutions: the associated Legendre functions of the first and the second kind respectively:

$$P_n^m(x) = \left(1-x^2\right)^{|m|/2} \frac{d^{|m|}}{dx^{|m|}} P_n(x) , \qquad (2.3)$$

$$Q_n^m(x) = \left(1-x^2\right)^{|m|/2} \frac{d^{|m|}}{dx^{|m|}} Q_n(x) \qquad (2.4)$$

Obviously the associated Legendre functions are not polynomials if m is odd.

The generating function

The *generating* function of the associated Legendre polynomials is given by:

$$\frac{(2m)!\left(1-x^2\right)^{m/2}}{2^m m!\left(1-2xt+t^2\right)^{m+1/2}} = \sum_{s=0}^{\infty} P_{s+m}^{m}(x)\, t^s \qquad (2.5)$$

$$|t| < 1$$

This form lacks of any direct physical application and it is seldom used

The Recurrence Relations

Associated Legendre polynomials obey the following recurrence relations:

- $$P_n^{m+1}(x) - \frac{2mx}{\left(1-x^2\right)^{1/2}} P_n^m(x) + \left[n(n+1) - m(m-1)\right] P_n^{m-1}(x) = 0$$
- $$(2n+1)xP_n^m(x) = (n+m)P_{n-1}^m(x) + \left[n-m+1\right]P_{n+1}^m(x) = 0$$
- $$(2n+1)\left(1-x^2\right)^{1/2} P_n^m(x) = P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x)$$
  $$= (n+m)(n+m-1)P_{n-1}^{m-1}(x) - (n-m+1)(n-m+2)P_{n+1}^{m-1}(x)$$
- $$\left(1-x^2\right)^{1/2}\left[P_n^m(x)\right]' = \frac{1}{2}P_n^{m+1}(x) - \frac{1}{2}\left(n+m\right)\left(n-m+1\right)P_n^{m-1}(x)$$

Associated Legendre Functions

Special Properties

- The Parity property:(with respect to $x = 0 \;, \theta = \pi/2$)

$$P_n^m(-x) = (-1)^{n+m} P_n^m(x) \qquad (2.7a)$$

$$P_n^m\left(\cos\left(\pi - \theta\right)\right) = (-1)^{n+m} P_n^m\left(\cos\left(\theta\right)\right) \qquad (2.7b)$$

- When Rodriguez formula (1.22) could be shown that it takes the form:

$$P_n^m(x) = \frac{1}{2^n n!}\left(1 - x^2\right)^{m/2}\left(\frac{d}{dx}\right)^{n+m}\left(x^2 - 1\right)^n \qquad (2.8)$$

Associated Legendre Functions

Orthogonality

Legendre's equation is self-adjoint. Which satisfies Sturm-Liouiville theory where the solutions are expected to be orthogonal to satisfying certain boundary conditions. Legendre polynomials are a set of orthogonal functions on (-1,1).

$$\int_{-1}^{1} P_p^m(x) P_q^m(x)\,dx = \frac{2}{2q+1}\cdot\frac{(q+m)!}{(q-m)!}\delta_{p,\,q} \qquad (2.9)$$

$$\int_{0}^{\pi} P_p^m(\cos\theta) P_q^m(\cos\theta)\sin\theta\, d\theta = \frac{2}{2q+1}\cdot\frac{(q+m)!}{(q-m)!}\delta_{p,\,q} \qquad (2.10)$$

$$\int_{-1}^{1} P_n^m(x) P_n^k(x)\left(1-x^2\right)^{-1} dx = \frac{(n+m)!}{m(n-m)!}\delta_{m,\,\mathrm{k}} \qquad (2.11)$$

Associated Legendre Functions

Series

- According to Sturm-Liouville theory that Legendre polynomial form a complete set.

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^m\left(x\right) \qquad (2.12)$$

- The coefficients a_n are obtained by multiplying the series by $P_m(x)$ and integrating in the interval [-1,1]

$$a_n = \frac{\left(2n+1\right)}{2\left(2n\right)!}\int_{-1}^{1} f(x) P_n^m(x)\, dx \qquad (2.13)$$

Useful notes

- The associated Legendre polynomials are the same for m and $-m$ provided that m^2 appears in d. e. (2.1).
- Given that a polynomial can be differentiated as many times as its degree then the inequality must hold: $|m| \leq n \Rightarrow -n \leq m \leq n$
- This means that for given n there are $2n+1$ different values of of m.

$$m = -n,\ -n+1,\ \ldots\ n-1,\ n$$