PHYS 404

Lecture 4: Bessel Functions

Dr. Vasileios Lempesis

Introduction

- Bessel functions appear in problems involving vibrations and heat conductions in regions with circular symmetry.
- They are named after the German mathematician and astronomer Friedrich Wilhelm Bessel who first used them to describe three body motion.
- They are solutions of the following differential equation:

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{\nu^2}{x^2}\right)u=0 \qquad (4.1)$$

The constant ν determines the order of the Bessel Functions and can take any real value. For cylindrical problems it is integer while for spherical is half-integer.

How they come up?

- If we try to solve the Helmholtz equation by separating variables in cylindrical coordinates then the radial equation is given by:

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\left[k^2-\frac{\nu^2}{\rho^2}\right]R(\rho)=0 \quad (4.2)$$

- With ν^2 the separation constant. Substitute $u(x)=R(\rho)$, with $x=k\rho$ and you will get again (4.1)

Solutions of Bessel Equation-a

- The solution of the Bessel Equation is given by: With the help of this function the most general solution of the Bessel equation can be written as

$$y(x) = AJ_{\nu}(x) + BJ_{-\nu}(x) \quad (4.3)$$

- The functions J_{ν}, $J_{-\nu}$ are called Bessel functions of first kind. They are given by:

$$J_{\nu}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s}}{s!\Gamma(\nu + s + 1)} \left(\frac{x}{2}\right)^{\nu+2s} \quad (4.4)$$

Solutions of Bessel Equation-b

- When ν is an integer then J_ν, $J_{-\nu}$ are not linearly independent since $J_{-n}(x) = (-1)^n J_n(x)$ thus we construct a second linearly independent solution the so called **Bessel function of second kind** or **Neumann function**:

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \qquad (4.5)$$

Bessel Functions of First Kind

Bessel functions of the first kind, denoted as J_ν , are solutions of Bessel's differential equation that are finite at the origin ($x = 0$) for integer or positive ν, and diverge as x approaches zero for negative non-integer ν. For integer $\nu=n$ it is possible to define the function by its Taylor series expansion around x = 0 as follows:

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} = \frac{x^n}{2^n n!} - \frac{x^{n+2}}{2^{n+2}(n+1)!} + \dots \qquad (4.6)$$

For this function it is easy to show that $J_{-n}(x) = (-1)^n J_n(x)$

Bessel functions of first kind are finite at x=0 for all real values of ν.

Bessel Functions of First Kind

The generating function

The Bessel functions can be obtained with the help of the so called *generating* function:

$$g(x,t) = e^{(x/2)(t-1/t)} \qquad (4.7)$$

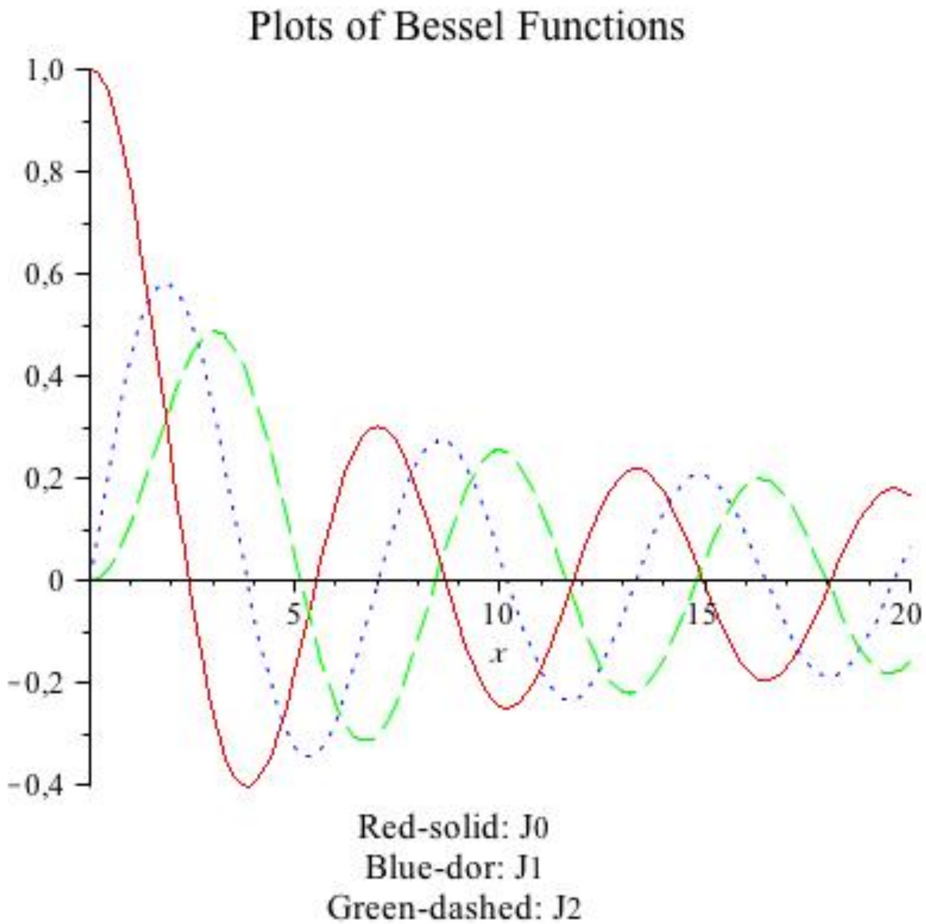
If we try to expand this function as a Laurent series we obtain

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n \qquad (4.8)$$

Hint: Generating function may define only Bessel functions of integral order.

Bessel Functions of First Kind

plots



Hint: The zeroes of the Bessel functions are not equidistant!

Bessel Functions of First Kind

The recurrence relations

For Bessel functions of first kind we can prove the following relations:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \qquad (4.9)$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n^{'}(x) \qquad (4.10)$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x) \qquad (4.11a)$$

$$\frac{d}{dx}\left[x^{n} J_n(x)\right] = x^{n} J_{n-1}(x) \qquad (4.11b)$$

Bessel Functions of First Kind

The integral representation function

Using the generating function we may prove a particularly useful and powerful way of representing the Bessel functions with the help of integrals.

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta, \qquad n = 0,\ 1,\ 2,\ 3,\ \ldots \tag{4.12}$$

Bessel Functions of First Kind

Orthogonality

If we divide Bessel's equation by x, it becomes self-adjoint and, therefore, by Sturm-Liouiville theory the solutions are expected to be orthogonal - if we can arrange to have appropriate boundary conditions satisfied. To take care of the boundary conditions for a finite interval [0, a], we introduce parameters a and $a_{\nu m}$ into the argument of J_ν *to get* $J_\nu(a_{\nu m}\rho/a)$. In this case we can show that

$$\int_0^a J_\nu\left(a_{\nu m}\frac{\rho}{a}\right) J_\nu\left(a_{\nu n}\frac{\rho}{a}\right)\rho\, d\rho = 0 \qquad (4.13)$$

$$\int_0^a \left[J_\nu\left(a_{\nu m}\frac{\rho}{a}\right)\right]^2 \rho\, d\rho = \frac{a^2}{2}\left[J_{\nu+1}\left(a_{\nu m}\right)\right] \qquad (4.14)$$

Bessel Functions of First Kind

Bessel series

- If we assume that the set of Bessel functions $J_\nu(\alpha_{\nu m}\rho / a)$ (ν fixed, $m = 1, 2, 3, \ldots$) is complete, then any well behaved but otherwise arbitrary function $f(\rho)$ may be expanded in a Bessel series (Bessel-Fourier series):

$$f(\rho) = \sum_{m=1}^{\infty} c_{mn} J_\nu(\alpha_{\nu m}\rho / a), \qquad 0 \le \rho \le a, \qquad \nu > -1 \qquad (4.15)$$

$$c_{\nu m} = \frac{2}{a^2 \left[J_{\nu+1}(\alpha_{\nu m})\right]^2} \int_0^a f(\rho) J_\nu(\alpha_{\nu m}\rho / a)\rho d\rho$$

Bessel Functions of First Kind

Potential Applications

- Problems involving electric fields, vibrations, heat conduction, optical diffraction plus others involving cylindrical or spherical symmetry.
- Transient heat conduction in a thin wall
- Steady heat flow in a circular cylinder of finite length.

Bessel Functions of Second Kind
Neumann Functions-a

The Neumann function is defined as follows:

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \qquad (4.16)$$

For nonintegral ν the above function clearly satisfies Bessel's equation, for it is a linear combination of known solutions. However if ν is integer the definition is given by

$$N_n(x) = \frac{1}{\pi}\left[\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial \nu}\right]_{\nu=n} \qquad (4.17)$$

Bessel functions of 2nd kind satisfy the same recurrence relations as the first kind.

Bessel Functions of Second Kind
Neumann Functions-b

- With the help of this function the most general solution of the Bessel equation can be written as

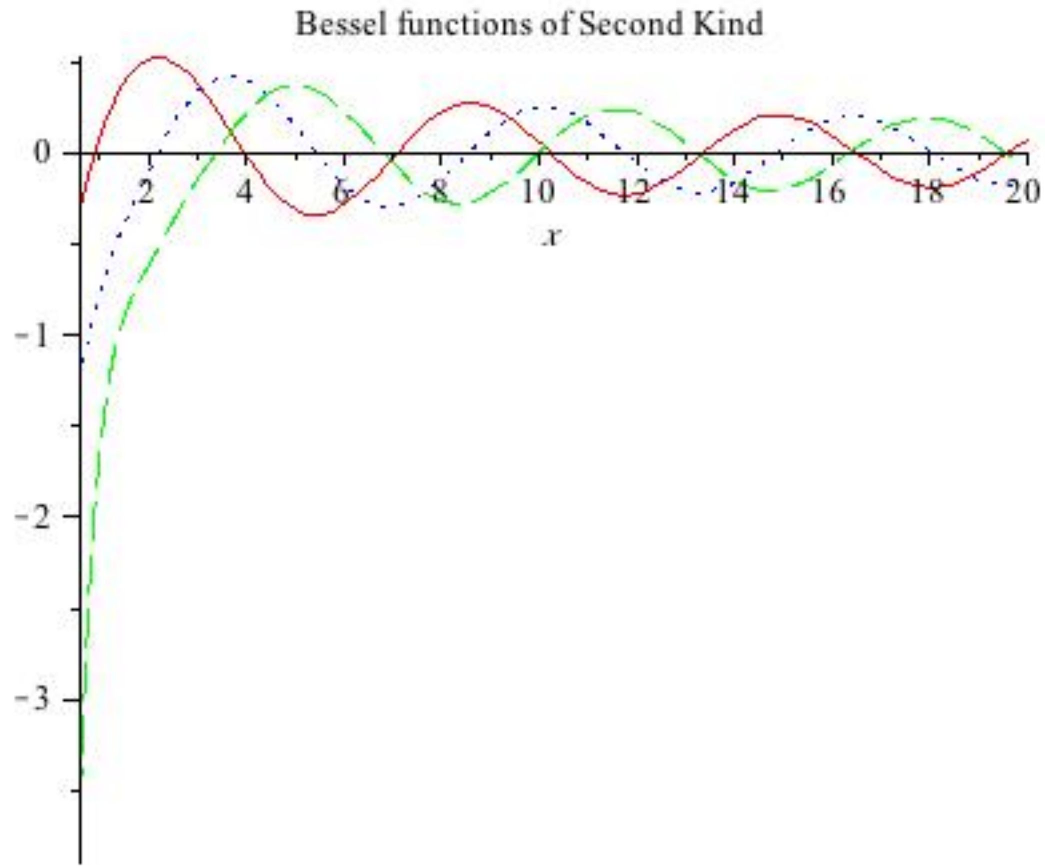
$$y(x) = AJ_\nu(x) + BN_\nu(x) \quad (4.18)$$

with A and B as arbitrary constants determined by boundary conditions.

- This is the only choice when ν *is an integer*. In this case we know that $J_{-n}(x) = (-1)^n J_n(x)$ thus the two Bessel functions are linearly dependent.
- When ν is non-integer the above equation is redundant and we may write

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x) \quad (4.19)$$

Bessel Functions of Second Kind



Solid Red: Y_0
Blue Dot: Y_1
Green Dash: Y_2