PHYS 404
Lecture 9: Fourier Transforms

Dr. Vasileios Lempesis

Introduction-a

- Frequently in mathematical physics we encounter pairs of functions related by an expression of the following form:

$$g(\alpha) = \int_a^b f(t)K(\alpha,t)dt$$

The function $g(\alpha)$ is called the **integral transform of** $f\left(t\right)$ by the kernel $K(\alpha,t)$.

Introduction-b

Examples of integral transforms

Integral Transforms	Kernel	Form
Fourier	e^{iat}	$\int_{-\infty}^{+\infty} f(t)e^{-iat}dt$
Laplace	e^{-at}	$\int_{-\infty}^{+\infty} f(t)e^{-at}dt$
Hankel	$tJ_n(at)$	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)tJ_n(\alpha t)dt$
Mellin	$t^{\alpha-1}$	$\int_{-\infty}^{+\infty} f(t)t^{\alpha-1}dt$

Introduction-c
Linearity

All these transforms are linear; that is

$$\int_a^b \left[c_1 f_1(t) + c_2 f_2(t)\right] K(\alpha, t)\,dt =$$

$$\int_a^b c_1 f_1(t) K(\alpha, t)\,dt + \int_a^b c_2 f_2(t) K(\alpha, t)\,dt$$

$$\int_a^b c f(t) K(\alpha, t)\,dt = c \int_a^b f(t) K(\alpha, t)\,dt$$

Introduction-c

The inverse operator

If we adopt for our linear integral transformation the operator *L*, we obtain

$$g(\alpha) = Lf(t)$$

For all the above transforms we have an inverse operator L^{-1} such that

$$f(t) = L^{-1}g(a)$$

The Fourier Transform

The problem that a Fourier transform answers is the representation of a non-periodic function over the infinite range. A physical example of this is the resolution of a wave packet into sinusoidal waves. The Fourier transform is defined by

$$g(\omega) = F\left[f(t)\right] = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt$$

$$f(t) = F^{-1}\left[g(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\omega)e^{i\omega t}d\omega$$

The Fourier Transform

The cosine transform

If the function is even it may be represented by the so called *Fourier cosine* transform.

$$g_c(\omega) = F\left[f(t)\right] = 2\int_{0}^{+\infty} f(t)\cos\left(\omega t\right)dt$$

$$f(t) = F^{-1}\left[g_c(\omega)\right] = \frac{1}{\pi}\int_{0}^{+\infty} g_c(\omega)\cos\left(\omega t\right)d\omega$$

The Fourier Transform

The sine transform

If the function is odd it may be represented by the so called *Fourier sine* transform.

$$g_s(\omega) = F\left[f(t)\right] = 2\int_0^{+\infty} f(t)\sin(\omega t)\,dt$$

$$f(x) = F^{-1}\left[g_s(\omega)\right] = \frac{1}{\pi}\int_0^{+\infty} g_s(\omega)\sin(\omega x)\,d\omega$$

$f(t)$	$g(\omega)$
$g(t - t_0)$	$g(\omega)e^{-i\omega t_0}$
$g(t)e^{i\omega_0 t}$	$g(\omega - \omega_0)$
$g(at)$	$(1/\lvert a\rvert)g(\omega/a)$
$g(-t)$	$g(-\omega)$
$g^{(n)}(t)$	$(i\omega)^n g(\omega)$
$\int_{-\infty}^{t} g(x)dx$ (If $\int_{-\infty}^{\infty} g(t)dt = G(0) = 0$)	$(1/i\,\omega)g(\omega)$

The Fourier Transform

In three dimensional space

When we move to three dimensional space the Fourier transform becomes:

$$g(\mathbf{k}) = \int f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3x$$

$$f(x) = \frac{1}{(2\pi)^3} \int g(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3k$$

The Fourier Transform

The convolution theorem-a

Some times in the probability theory we need to determine the probability density of two random independent variables *f* and *g*. This is given by the so called *convolution* of the functions;

$$f(t) * g(t) \equiv \int_{-\infty}^{\infty} f(x) g(t-x) dt$$

$$f(t) * g(t) = g(t) * f(t), \quad [f_1(t) * f_2(t)] * f_3(t) = f_1(t) * [f_2(t) * f_3(t)]$$

$$f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$$

The Fourier Transform

The convolution theorem-b

Let's assume that $J(t)$ and $G(t)$ are the Fourier transforms of the functions j and g. It can be proved that the Fourier inverse transform of a *product* of Fourier transforms is the convolution of the original functions

$$F\left[j(t) * g(t)\right] = J(\omega)G(\omega) \quad F^{-1}\left[J(\omega)G(\omega)\right] = j(t) * g(t)$$

The Fourier Transform

The frequency convolution theorem

Let the functions $f_1(t)$, $f_2(t)$ with Fourier transforms $F_1(\omega)$, $F_2(\omega)$. We can prove the *frequency convolution* theorem:

$$F^{-1}\left[F_1(\omega)*F_2(\omega)\right]=2\pi f_1(t)f_2(t)$$

Let's assume that $F(\omega)$ is the Fourier transform of the function f . It can be proved

$$\int_{-\infty}^{\infty}\left|f(t)\right|^2 dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(\omega)\right|^2 d\omega$$

The Fourier Transform

The momentum representation-a

In quantum mechanics the wave function, $\psi(x)$, which is a solution of Schrödinger equation, has the following properties:

1. $\psi^*(x)\psi(x)dx$ is the probability of finding the particle between x and $x+dx$.

2. $$\int_{-\infty}^{+\infty} \psi^*(x)\psi(x)dx = 1$$

3. $$\langle x \rangle = \int_{-\infty}^{+\infty} \psi^*(x)x\psi(x)dx$$

The Fourier Transform

The momentum representation-b

We want a function $g(p)$, that will give the same information for momentum:

1. $g^*(p)g(p)dp$ is the probability of finding the particle with momentum between p and $p+dp$.
2. $$\int_{-\infty}^{+\infty} g^*(p)g(p)dp = 1$$
3. $$\langle p \rangle = \int_{-\infty}^{+\infty} g^*(p)\,p\,g(p)dp$$

The Fourier Transform
The momentum representation-c

Such a function is given by the Fourier transform of our space function $\psi(x)$,

$$g(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar} dx$$

$$g^*(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi^*(x) e^{ipx/\hbar} dx$$

The corresponding three-dimensional momentum function is

$$g(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \iiint_{-\infty}^{\infty} \psi(\mathbf{r}) e^{-i\mathbf{r}\cdot\mathbf{p}/\hbar} d^3x$$

Mathematical Supplement
The Dirac Delta function -a

- The Dirac function in the one-dimensional case is defined as:

$$\delta(x) = 0, \quad x \neq 0$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1, \qquad \int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

Mathematical Supplement
The Dirac Delta function -b

- Delta function has also the following properties:

$$\delta\left[a\left(x-x_1\right)\right]=\frac{1}{a}\delta\left(x-x_1\right),$$

$$\delta\left[\left(x-x_1\right)\left(x-x_2\right)\right]=\left[\delta\left(x-x_1\right)+\delta\left(x-x_2\right)\right]/\left|x_1-x_2\right|$$

$$x\frac{d\delta\left(x\right)}{dx}=-\delta\left(x\right),\quad \int_{-\infty}^{\infty}\delta'\left(x\right)f(x)dx=-f'(0)$$

$$F\left[\delta\left(t\right)\right]=1,\quad F(e^{i\omega_0 t})=2\pi\delta\left(\omega-\omega_0\right)$$

The Fourier Transform

…of a periodic function

Periodic functions can also be Fourier transformed. We can show that if $g(t)$ is a periodic function with period T then its Fourier transform is given by

$$F(\omega) = 2\pi \sum_{n=-\omega}^{\infty} c_n \delta\left(\omega - n\omega_0\right)$$

Where the coefficients c_n are associated with the corresponding Fourier series representation of the function

Mathematical Supplement

Tables of Fourier transforms

In the literature we can find tables with Fourier transforms of different functions. We give here such a table which also contains useful trigonometric formulae and integrals. (Click here).