Lecture (14)

Ch 5

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Ex (5.6) p. (65)

In the valuation of warranties on automobiles, it is important to recognize that the number of miles driven varies from driver to driver. It is also the case that for a particular driver the number of miles varies from year to year. Suppose that the number of miles for a randomly selected driver has an inverse Weibull distribution but the year-to-year variation in the scale parameter has a transformed gamma distribution with the same value for τ. Determine the distribution for the number of miles driven in a randomly selected year by a randomly selected driver.

Ans: let X # miles covered by a driver

$X|\Lambda$ # miles covered in a year by a driver

$$X|\Lambda \sim \text{inv Weibull } (\lambda, \tau)$$

where Λ is the scale parameter random variable

$$\Lambda \sim \text{transformed gamma } (\alpha, \theta, \tau)$$

The year-to-year variation

$$f_{X|\Lambda}(x|\lambda) = \frac{\tau(\lambda/x)^{\tau} e^{-(\lambda/x)^{\tau}}}{x}$$

$$f_{X|\Lambda}(x|\lambda) = \frac{\tau \lambda^{\tau} e^{-(\lambda/x)^{\tau}}}{x^{\tau+1}} \quad (1)$$

For inv. Weibull

$$f(x) = \frac{\tau(\theta/x)^{\tau} e^{-(\theta/x)^{\tau}}}{x}$$

See p. (469)

$$f_{\Lambda}(\lambda) = \frac{\tau}{\theta^{\tau\alpha}\Gamma(\alpha)} \lambda^{\tau\alpha-1} e^{-\lambda^{\tau}\theta^{-\tau}} \quad (2)$$

For transformed gamma

$$f(x) = \frac{\tau(x/\theta)^{\tau\alpha} e^{-(x/\theta)^{\tau}}}{x\Gamma(\alpha)}$$

See p. (467)

$$\therefore f_X(x) = \int f_{X|\Lambda}(x|\lambda) f_{\Lambda}(\lambda)\, d\lambda \quad (3)$$

Substitute ①, ② in ③, we get

$$f_X(x) = \frac{\tau^2}{\theta^{\tau\alpha}\Gamma(\alpha)\, x^{\tau+1}} \int_0^\infty \lambda^{\tau+\tau\alpha-1} \exp\left[-\lambda^\tau(x^{-\tau}+\theta^{-\tau})\right] \cdot d\lambda$$

let $y = \lambda^\tau(x^{-\tau}+\theta^{-\tau})$

$$\Rightarrow \lambda = y^{1/\tau}(x^{-\tau}+\theta^{-\tau})^{-1/\tau}$$

$$\therefore f_X(x) = \frac{\tau^{\cancel{2}}}{\theta^{\tau\alpha}\Gamma(\alpha)\, x^{\tau+1}} \int_0^\infty \left[y^{1/\tau}(x^{-\tau}+\theta^{-\tau})^{-1/\tau}\right]^{\tau+\tau\alpha-1} \cdot e^{-y} \cdot \cancel{\frac{1}{\tau}}\, y^{1/\tau-1}(x^{-\tau}+\theta^{-\tau})^{-1/\tau}\, dy$$

$$\therefore f_X(x) = \frac{\tau}{\theta^{\tau\alpha}\Gamma(\alpha)\, x^{\tau+1}(x^{-\tau}+\theta^{-\tau})^{\alpha+1}} \int_0^\infty y^\alpha e^{-y}\, dy$$

$$\therefore f_X(x) = \frac{\tau\, \Gamma(\alpha+1)}{\theta^{\tau\alpha}\Gamma(\alpha)\, x^{\tau+1}(x^{-\tau}+\theta^{-\tau})^{\alpha+1}}$$

Note

$$(x^{-\tau}+\theta^{-\tau})^{-1-\alpha+1/\tau} \cdot (x^{-\tau}+\theta^{-\tau})^{-1/\tau} = (x^{-\tau}+\theta^{-\tau})^{-1-\alpha}$$

$$\therefore f_X(x) = \frac{\tau\alpha\, \cancel{\Gamma(\alpha)}}{\theta^{\tau\alpha}\cancel{\Gamma(\alpha)}\, x^{\tau+1}(x^{-\tau}+\theta^{-\tau})^{\alpha+1}}$$

$$\therefore f_X(x) = \frac{\tau\alpha\, (x\theta)^{\tau(\alpha+1)}}{\theta^{\tau\alpha}\, x^{\tau+1}(x^{\tau}+\theta^{\tau})^{\alpha+1}}$$

$$\therefore f_X(x) = \frac{\tau\alpha\, \theta^{\cancel{\tau\alpha}+\tau}\, x^{\tau\alpha+\cancel{\tau}}}{\cancel{\theta^{\tau\alpha}}\, x^{\cancel{\tau}+1}(x^{\tau}+\theta^{\tau})^{\alpha+1}}$$

Notes

$$y^{1+\alpha-1/\tau} \cdot y^{1/\tau-1} = y^\alpha$$

$$(x^{-\tau}+\theta^{-\tau})^{\alpha+1} = \left(\frac{1}{x^\tau}+\frac{1}{\theta^\tau}\right)^{\alpha+1}$$

$$\therefore f_X(x) = \frac{\tau\alpha\,\theta^\tau x^{\tau\alpha-1}}{(x^\tau+\theta^\tau)^{\alpha+1}}$$

which is the pdf of inverse Burr distn (τ, θ, α).

See p. 464