Lecture (15)

• Sec 5.2.5 Frailty Models p. (65)

let $\Lambda > 0$ be frailty random variable and define the conditional hazard rate

$$h_{X|\Lambda}(x|\lambda) = \lambda\, a(x),$$

$a(x)$ is known function for a certain application.

$\Rightarrow$ the conditional survival fn of $X|\Lambda$ is therefore

$$S_{X|\Lambda}(x|\lambda) = e^{-\int_0^x h_{X|\Lambda}(t|\lambda)\,dt}$$

$$= e^{-\lambda \int_0^x a(t)\,dt}$$

$$\therefore S_{X|\Lambda}(x|\lambda) = e^{-\lambda A(x)},$$

where $A(x) = \int_0^x a(t)\,dt$

* In order to specify the mixture distribution (i.e. the marginal distn of X), we define the moment generating fn of the frailty random variable Λ to be

$$M_\Lambda(z) = E\left(e^{z\Lambda}\right).$$

• Then, the marginal survival function is

$$S_X(x) = E\left[e^{-\Lambda A(x)}\right] = M_\Lambda\left[-A(x)\right]$$

$$\Rightarrow \quad F_X(x) = 1 - M_\Lambda\left[-A(x)\right]$$

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Ex 5.7 p. 66

Let Λ have a gamma distribution and let $X|\Lambda$ have a weibull distribution with Conditional survival function $S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$. Determine the unconditional or marginal distn of X.

Ans: let $\Lambda \sim$ gamma (θ, α), $X|\Lambda \sim$ Weibull (λ, γ)

See p. 469

$$\because S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$$

$$\therefore \boxed{A(x) = x^{\gamma}}$$

$$\because S_X(x) = E\left[e^{-\Lambda A(x)}\right]$$

$$= M_\Lambda[-A(x)]$$

$$\therefore S_X(x) = M_\Lambda(-x^{\gamma})$$

and $\because M_\Lambda(z) = (1 - \theta z)^{-\alpha}$

$$\therefore S_X(x) = (1 + \theta x^{\gamma})^{-\alpha}$$

which is a Burr distn with parameters

$\theta \longrightarrow \theta^{-1/\gamma}$, $\alpha \longrightarrow \alpha$ (not change) See p. 464

Revise Ex 1, lecture 6

For $X \sim$ gamma (θ, α)

$M_X(z) = E(e^{zX})$

$= \left(\frac{1}{1-\theta z}\right)^{\alpha}$, $z < \frac{1}{\theta}$

- Note that when $\gamma = 1$, this is an exponential mixture which is a pareto distn, with parameters $\theta \to 1/\theta$, $\alpha \to \alpha$

See p. 465

- **Sec 5.2.6 Splicing** ~ "Spliced distribution"

See p. 66

Def$^{\underline{n}}$ A K-component spliced distribution has a density function that can be expressed as follows:

$$f_X(x) = \begin{cases} a_1 f_1(x), & c_0 < x < c_1 \\ a_2 f_2(x), & c_1 < x < c_2 \\ \vdots & \vdots \\ a_k f_k(x) & c_{k-1} < x < c_k \end{cases}$$

For $j = 1, \dots, k$, each $a_j > 0$ and each $f_j(x)$ must be legitimate density function with all probability on the interval (c_{j-1}, c_j). Also, $a_1 + \dots + a_k = 1$.

Ex 5.8 p. 67

Demonstrate that Model 5 in Section 2.2 is a two-component spliced model.

Ans:

For Model 5, the density fn is Revise Lecture 3

$$f_X(x) = \begin{cases} 0.01, & 0 \le x < 50 \\ 0.02, & 50 \le x < 75 \end{cases}$$

$$\Rightarrow f_X(x) = \begin{cases} 0.5(0.02), & 0 \le x < 50 \\ 0.5(0.04) & 50 \le x < 75 \end{cases} = \begin{cases} a_1 f_1(x), & 0 \le x < 50 \\ a_2 f_2(x), & 50 \le x < 75 \end{cases}$$

* It is a spliced model created by letting $f_1(x) = 0.02$, $0 \le x < 50$ which is a uniform dist$^{\underline{n}}$ on the interval from 0 to 50, and $f_2(x) = 0.04$, $50 \le x < 75$, which is a uniform dist$^{\underline{n}}$ on the interval from 50 to 75. The coefficients are $a_1 = 0.5$ and $a_2 = 0.5$.

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