48) $\begin{cases} y^{(4)} = 0 \\ y(0) = 2\ ,\ y'(0) = 3\ , y''(0) = 4,\ \ y'''(0) = 5 \end{cases}$

49) $\begin{cases} y^{(4)} - 3y''' + 3y'' - 3y' = 0 \\ y(0) = y'(0) = 0\ \ ,\ y''(0) = y'''(0) = 1 \end{cases}$

50) $\begin{cases} y^{(4)} - y = 0 \\ y(0) = y'(0) = y''(0) = 0\ ,\ \ y'''(0) = 1. \end{cases}$

51) a) Show that the solution of the(IVP)

$$\begin{cases} y'' - 2ry' + (r^2 - \frac{\alpha^2}{4})y = 0 \\ y(0) = 0\ ,\ y'(0) = 1. \end{cases}$$

is given by

$$y_\alpha(x) = \frac{1}{\alpha}\left[e^{(r+\frac{\alpha}{2})x} - e^{(r-\frac{\alpha}{2})x}\right],$$

where r and α are real positive .

b) Show that $\lim\limits_{\alpha \to 0} y_\alpha(x) = xe^{rx}$.

4.4 Cauchy- Euler Differential equation

A Cauchy Euler homogeneous differential equation is of the form

$$a_n x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + ... + a_1 xy' + a_0 y = 0 \qquad (1)$$

where a_n ,a_{n-1} ,....., a_1, a_0 are constants and $a_n \neq 0$. Since the leading coefficient $a_n x^n$ should never be zero, the interval of definition of the differential equation (1) is either the open interval $(0, \infty)$ or $(-\infty, 0)$. That is, the differential equation (1) should be solved for either $x > 0$ or $x < 0$.

The *Euler* differential equation is probably the simplest type of linear differential equations with variable coefficients. The reason for this is that the change of independent variable

$$x = \begin{cases} e^t\ ;\ x > 0 \\ -e^t\ ;\ \ x < 0 \end{cases},$$

produces a differential equation with constant coefficients. We illustrate this fact for a second -order case.

Example(1) Show by means of change of independent variable above that the *Euler* differential equation of second order

$$a_2 x^{2'} y'' + a_1 xy' + a_0 y = 0, \qquad (2)$$

with a_0 , a_1 and a_2 are given constants, is reduced to the differential equation

$$a_2y''(t) + (a_1 - a_2)y'(t) + a_0y = 0. \tag{3}$$

Proof First we assume that $x > 0$. Then the transformation $x = e^t$ implies that

$$t = \ln x, \quad y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = y'(t)\frac{1}{x},$$

hence

$$xy\prime = y\prime(t)$$

Now

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\,(y'(t)\frac{1}{x}) = -\frac{1}{x^2}y'(t\,) + y''(t\,)\frac{1}{x^2}\,.$$

Then

$$x^2y'' = y''(t) - y'(t).$$

Substituting xy' and $x^2\,y''$ into the differential equation (2), we obtain :

$$a_2x^{2'}y'' + a_1xy' + a_0y = a_2((y''t) - y'(t)) + a_1y'(t) + a_0y = 0,$$

$$a_2y''(t) + (a_1 - a_2)y'(t) + a_0y = 0.$$

Now if we assume that $x < 0$. we set $x = -e^t$, so $\ln(-x) = t$ and we have also

$$xy' = y'(t) \quad , \; x^2y'' = y''(t) - y'(t),$$

and

$$a_2y''(t) + (a_1 - a_2)y'(t) + a_0y = 0.$$

Hence the equation (3) is true in two cases $x > 0$ and $x < 0$.

Example (2) From the example (1) , discuss the general solution of the *Euler* differential equation of second order 2.

Solution First we find the general solution of the differential equation (3). In fact the characteristic equation of (3)

$$a_2m^2 + (a_1 - a_2)m + a_0 = 0. \tag{4}$$

Thus we have

$$y(t) = \begin{cases} c_1e^{m_1t} + c_2e^{m_2t} & \text{if } m_1 \text{ and } m_2 \text{ are real such that } m_1 \neq m_2 \\ (c_1 + c_2t)e^{mt} & \text{if } m_1 \text{ and } m_2 \text{ are real such that } m_1 = m_2 = m \\ c_1e^{\alpha t}\cos(\beta t) + c_2e^{\alpha t}\sin(\beta t) & \text{if } m_1 = \alpha + \beta i \;, m_2 = \alpha - \beta i \;; \;\; \beta \neq 0 \end{cases}$$

Now for $x > 0$, $x^m = e^{mt}$, $t = \ln x$. Thus the general solution of the *Euler* differential equation (2) has the form

$$y(x) = \begin{cases} c_1 x^{m_1} + c_2 x^{m_2} & \text{if } m_1 \text{ and } m_2 \text{ are real such that } m_1 \neq m_2 \\ (c_1 + c_2 \ln x) x^m & \text{if } m_1 \text{ and } m_2 \text{ are real such that } m_1 = m_2 = m \\ c_1 x^\alpha \cos(\beta \ln x) + c_2 x^\alpha \sin(\beta \ln x) & \text{if } m_1 = \alpha + \beta i \ , m_2 = \alpha - \beta i \ ; \ \beta \end{cases}$$

Now if $x < 0$, then $-x = e^t$ and x should be replaced by $-x$ in the formula for $y(x)$.

Example (3) Solve the *Euler* differential equation

$$2x^2 y'' - 3xy' - 3y = 0. \tag{5}$$

a) For $x > 0$, b) For $x < 0$.

Solution a) Using the transformation $x = e^t$, the differential equation becomes (see the equation (3))

$$2y''(t) - 5y'(t) - 3y(t) = 0. \tag{6}$$

The characteristic equation for this differential equation is

$$2m^2 - 5m - 3 = 0.$$

So the roots of this equation are $m_1 = -\frac{1}{2}$, $m_2 = 3$ .Thus

$$y(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{3t},$$

is the general solution of the equation (6), hence

$$y(x) = c_1 x^{-\frac{1}{2}} + c_2 x^3.$$

is the general solution of (5).

b) Using the transformation $-x = e^t$, from part **a)**, we deduce that

$$y(x) = c_{1(-x)}^{-\frac{1}{2}} + c_2(-x)^3.$$

is the general solution of the equation (5).

The results of the example 1 and 2 are associated with the *Euler* differential equation of order 2. But to solve any *Euler* differential equation, we try a substitution of the form $y = x^m$, where $x > 0$ and m is a root of a polynomial equation.

Example (4) Solve the *Euler* differential equation :

$$x^3y''' - x^2y'' - 2xy' - 4y = 0 \quad , \quad \text{for } x > 0. \tag{7}$$

Solution We substitute $y = x^m$ in the differential equation (7) we obtain

$$[m(m-1)(m-2) - m(m-1) - 2m - 4]\, x^4 = 0,$$

since $x^m \neq 0$, then we have

$$m^3 - 4m^2 + m - 4 = m^2(m-4) + (m-4) = (m^2+1)(m-4) = 0.$$

Hence $m = \mp i$, $m = 4$. we can prove that x^4 , $\cos(\ln x)$ and $\sin(\ln x)$ are linearly independent on $(0, \infty)$. Therefore the general solution of (7) is

$$y = c_1x^4 + c_2\cos(\ln x) + c_3\sin(\ln x).$$

Now we substitute $y = x^m$ in the equation (1) , for $x > 0$

$$a_nx^ny^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \ldots\ldots + a_1xy' + a_0y = 0. \tag{1}$$

Then we have

$$y' = mx^{m-1} \cdot y'' = m(m-1)x^{m-2}, \ldots\ldots\ldots, \; y^{(n)} = m(m-1)(m-2)\ldots.(m-n+1)x^{m-n},$$

and

$$x^m\,[a_nm(m-1)(m-2)\ldots.(m-n+1) + a_{n-1}m(m-1)(m-2)\ldots.(m-n+2) + \ldots. \\ +a_2m(m-1) + a_1m + a_0] = 0.$$

Since $x^m \neq 0$, we find

$$a_nm(m-1)(m-2)\ldots.(m-n+1)+a_{n-1}m(m-1)(m-2)\ldots.(m-n+2)+\ldots.+a_2m(m-1)+a_1m+a_0 \tag{8}$$

1) If $m = m_i$ is a root of multiplicity k $(k \leq n)$ of (8), then we can prove that the functions

$$x^m, x^m(\ln x), x^m(\ln x)^2, \ldots\ldots, x^m(\ln x)^{k-1}$$

are solutions of (1) linearly independent on $(0, \infty)$.

2) In the equation (1) we suppose that a_0 , a_1 , a_2 ,.....,a_n are real numbers. So if (8) has complex root $m = \alpha + \beta i$ of multiplicity k , then the equation

(8) has the complex conjugate $m = \alpha - \beta i$ as root again of multiplicity k. We prove that the functions

$$x^{\alpha}\cos(\beta\ln x) \ , \ x^{\alpha}\sin(\beta\ln x) \ , \ x^{\alpha}(\ln x)\cos(\beta\ln x) \ , \ x^{\alpha}(\ln x)\sin(\beta\ln x) \ , \ ...., \\ x^{\alpha}(\ln x)^{k-1}\cos(\beta\ln x) \ , \ x^{\alpha}(\ln x)^{k-1}\sin(\beta\ln x).$$

are solutions of the differential equation (1) linearly independent on $(0,\infty)$, where $2k \leq n$.

3) If m_1 ,m_2 , ...,m_k are real roots of (8) such that $m_i \neq m_j$ then we can prove that the functions

$$x^{m_1}, x^{m_2}, ..., x^{m_k}$$

are solutions of the differential equation (1) and that they are linearly independent on $(0,\infty)$. We can prove these results at least when $n = 2$.

Example (5) Find the general of the differential equation

$$x^2y'' - 3xy' + 13y = 0 \quad ; \quad x > 0. \tag{9}$$

Solution Substituting $y = x^m$ in the equation (9), we obtain

$$m(m-1) - 3m + 13 = m^2 - 4m + 13 = 0.$$

Then we have two complex roots $m = 3 \mp 3i$, hence the the general of the differential equation (9) is

$$y = c_1x^3\cos(3\ln x) + c_2x^3\sin(3\ln x) \quad ; \quad x > 0.$$

If we suppose $x < 0$, then the general of the differential equation (9) is

$$y = c_1(-x)^3\cos(3\ln(-x)) + c_2(-x)^3\sin(3\ln(-x)) \quad ; \quad x < 0.$$

Example (6). Find the general solution of the differential equation

$$x^4y^{(4)} - 5x^3y''' + 3x^2y'' - 6xy' + 6y = 0 \quad ; \quad x > 0. \tag{10}$$

Solution Substituting $y = x^m$ in the equation (10), we obtain

$$m(m-1)(m-2)(m-3) - 5m(m-1)(m-2) + 3m(m-1) - 6m + 6 = 0.$$

This implies that

$$(m-1)(m-2)(m^2 - 8m + 3) = 0.$$

The roots of this equation are $m = 1$, $m = 2$, and $m = 4 \mp \sqrt{13}$, then the general solution of the differential equation (10) is

$$y = c_1 x + c_2 x^2 + c_3 x^{4+\sqrt{13}} + c_4 x^{4-\sqrt{13}} \quad ; \quad x > 0.$$

Example (7) Find the general solution of the differential equation

$$x^5 y^{(5)} - 2x^3 y''' + 4x^2 y'' = 0 \quad ; \quad x < 0. \tag{11}$$

Solution Substituting $y = x^m$ in the equation (11) we obtain

$$m(m-1)(m-2)(m-3)(m-4) - 2m(m-1)(m-2) + 4m(m-1) = 0,$$

$$m(m-1)(m^3 - 9m^2 + 24m - 20) = m(m-1)(m-2)^2(m-5) = 0.$$

So the roots of this equation are $m = 0$, $m = 1$, $m = 2$ repeated two times and $m = 5$, then the general of the differential equation (11) is

$$y = c_1 + c_2(-x) + c_3(-x)^2 + c_4(-x)^2 \ln(-x) + c_5(-x)^5.$$

Example (8) Find the general solution of the differential equation :

$$(x-4)y'' - y' + \frac{1}{(x-4)}y = 0 \quad ; \quad x > 4. \tag{12}$$

Solution The equation (12) is also *Euler* differential equation , because

$$(x-4)^2 y'' - (x-4)y' + y = 0 \quad , \quad x > 4.$$

Then the solution of (12) is the form

$$y = (x-4)^m ,$$

hence

$$[m(m-1) - m + 1]\,(x-4)^m = 0.$$

But

$$(x-4)^m \neq 0,$$

then we have

$$(m-1)^2 = 0 \ ,$$

Consequently $m = 1$ is a root repeated two times, so the general solution of the differential equation (12) is

$$y = c_1(x-4) + c_2(x-4)\ln(x-4).$$

Example (9) Solve the initial value problem (IVP)

$$\begin{cases} x^2y'' - 4xy' + 6y = 0 \\ y(-2) = 8 \ , \ y'(-2) = 0 \end{cases}.$$

Solution From the initial conditions. we should suppose $x < 0$ .The solution of the differential equation is the form $y = x^m$, hence

$$m(m-1) - 4m + 6 = (m-2)(m-3) = 0.$$

So we have two roots $m = 2$ and $m = 3$, so the general solution of the differential equation is

$$y = c_1(-x)^2 + c_2(-x)^3 = c_1x^2 + c_3x^3 \quad \text{where} \quad c_3 = -c_2.$$

From the initial conditions we have

$$c_1 - 2c_3 = 2,$$

and

$$-c_1 + 3c_3 = 0,$$

hence $c_1 = 6$ and $c_3 = 2$. Then the solution of the IVP is

$$y = 6x^2 + 2x^3.$$

Exercises(4.4)

In problems 1 through 20 find the general solution of the following differential equations , where we suppose that $x > 0$.

1) $x^2y'' - y = 0.$

2) $xy'' + y' = 0.$

3) $xy'' - y' = 0.$

4) $4x^2y'' + y = 0.$

5) $x^2y'' + 5xy' + 3y = 0.$

6) $x^2y'' + xy' + 4y = 0.$

7) $x^2y'' - 3xy' - 2y = 0.$

8) $25x^2y'' + 25xy' + y = 0.$

9) $4x^2y'' + 4xy' - y = 0.$

10) $x^2y'' + 5xy' + 4y = 0.$

11) $x^2y'' - xy' + 2y = 0.$

12) $x^2y'' + 8xy' + 6y = 0.$

13) $x^2y'' - 7xy' + 41y = 0.$

14) $3x^2y'' + 6xy' + y = 0.$

15) $2x^2y'' + xy' + y = 0.$

16) $x^3y'''' - 6y = 0.$

17) $x^3y'''' + xy' - y = 0.$

18) $x^3y'''' - 2x^2y'' - 2xy' + 8y = 0.$

19) $x^3y'''' - 2x^2y'' + 4xy' - 4y = 0.$

20) $x^3y'''' + 4x^2y'' - 8xy' + 8y = 0.$

21) In problems 5 through 12, solve the given differential equations by the substitution $x = e^t$.

In problems 22 through 30 find the solution of the initial values problems.

22) $\begin{cases} x^2y'' + 3xy' = 0 \\ y(1) = 0 \ , \ y'(1) = 4. \end{cases}$

23) $\begin{cases} x^2y'' + xy' + y = 0 \\ y(1) = 1 \ , \ y'(1) = 2. \end{cases}$

24) $\begin{cases} x^2y'' - 5xy' + 8y = 0 \\ y(2) = 32 \ , y'(2) = 0. \end{cases}$

25) $\begin{cases} x^2y'' - 3xy' + 4y = 0 \\ y(1) = 5 \ , \ y'(1) = 3. \end{cases}$

26 $\begin{cases} 4\,x^2y'' + y = 0 \\ y(-1) = 2, \ y'(-1) = 4. \end{cases}$

27) $\begin{cases} x^2y'' - 4xy' + 6y = 0 \\ y(-2) = 8 \ , \ y'(-2) = 0. \end{cases}$

28) $\begin{cases} x^2y'' - xy' + y = 0 \\ y(-1) = 1 \ , \ y'(-1) = 0. \end{cases}$

29) $\begin{cases} x^2y'' + \frac{7}{2}xy' - \frac{3}{2}y = 0 \\ y(-4) = 1 \ , y'(-4) = 0 \end{cases}$

30) $\begin{cases} x^3y'''' + 4x^2y'' - 8xy' + 8y = 0 \\ y(1) = 0\ ,\ y'(1) = 1\ , y''(1) = 0. \end{cases}$

The equation (8) is called also the characteristic equation corresponding to the *Euler* differential equation (1). Now in exercises 31 through 38, write only the characteristic equation associated with the *Euler* differential equation.

31) $3x^3y'''' - x^2y'' + 4xy' - 4y = 0 \quad , \quad x > 0.$

32) $x^4y^{(4)} - 5x^3y'''' + 3x^2y'' - 6xy' + 6y = 0 \quad , \quad x > 0.$

33) $2\,x^4y^{(4)} + 3x^3y'''' - 4x^2y'' + 8xy' - 8y = 0 \quad , \quad x < 0.$

34) $2x^3y'''' + x^2y'' - 12xy' - 2y = 0 \quad , \quad x < 0.$

35) $x^5y^{(5)} - 2x^3y'''' + 4x^2y'' = 0 \quad , \quad x < 0.$

36) $7x^4y^{(4)} - 2x^3y'''' + 3x^2y'' - 6xy' + 6y = 0 \quad , \quad x > 0.$

37) $x^5y^{(5)} + 2x^3y'''' - 9x^2y'' + 18x\ y' - 18y = 0\ , \quad x > 0.$

38) $x^6y^{(6)} - 12x^4y^{(4)} = 0 \quad , \quad x > 0.$

39) Astronomy *Kopal* obtained a differential equation of the form

$$r\psi''(r) + \psi'(r) = 0 \ \ , \ \ r > 0.$$

Solve the differential equation (Multiplication of the *D.E.* by r reduces an *Euler* differential equation).

40) FLOWS Borton and Rayor in the steady of peristaltic flow in tubes, obtained the following linear homogeneous differential equation with the form

$$P''(r) + \frac{1}{r}P'(r) = 0 \quad , r > 0.$$

Solve this differential equation (Multiplication of the *D.E.* by r^2 produces an *Euler* differential equation)

In Exercises 41 through 45 , find the general solutions of the differential equations .

41) $(x-1)^2y'' + 5(x-1)y' + 4y = 0 \quad ; \quad x > 1.$

42) $(x+3)y'' + 3(x+3)y' + 5y = 0 \ ; \ x < -3.$

43) $(x-2)^2y'' - (x-2)y' + y = 0 \quad ; \ x > 2.$

44) $(3x+4)y'' + 10(3x+4)y' + 9y = 0 \quad ; \quad 3x + 4 > 0.$

45) $(x+2)^2y'' + (x+2)y' + y = 0 \quad ; \quad x < -2\ .$

4.5 General solution of Nonhomogeneous linear differential equations

Let

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots\dots + a(x)y' + a_0(x)y = g(x). \qquad (1)$$

be a nonhomogeneous linear differential equation, where a_n , a_{n-1}, ...,a_1 , a_0 and g are continuous functions defined on an interval $I = (a\ ,b)$ such that $a_n(x) \neq 0$ for all $x \in I$ and g is not identically zero on I .

Let $\{y_1,\ y_2, ..., y_n\}$,be a fundamental set of solutions of homogeneous equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a(x)y' + a_0(x)y = 0. \qquad (2)$$

Now suppose that y_p is any particular solution of (1) and y_c is a solution of (2), then

$$y = y_c + y_p \qquad (3)$$

is a solution of (1). For using the equation (3) we see that

$$\begin{aligned} y &= a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a(x)y' + a_0(x)y, \\ &= (a_n(x)y_c^{(n)} + ... + a_1(x)y_c' + a_0(x)y_c) + (a_n(x)y_p^{(n)} + ... + a_1(x)y_p' + a_0(x)y_p), \\ &= 0 + g(x). \end{aligned}$$

But y_c is a solution of (2) , then there exist n constants c_1 . c_2, ..., c_n such that

$$y_c = c_1y_1 + c_2y_2 + ... + c_ny_n,$$

hence the solution of (1) is the form

$$y = c_1y_1 + c_2y_2 + ... + c_ny_n + y_p. \qquad (4)$$

The function y equation (4) is called the general solution of the differential equation (1).

Nonhomogeneous equations

(4.5.1)Undetermined coefficients

Various method for getting a particular solution of (1), where the coefficients a_0 ,a_1 , ..., $a_n \in \mathbb{R}$ such that ,$a_n \neq 0$. In preparation for the method of Undetermined coefficients. We suppose that g is a linear combination of functions of the following types.

1) $C\ x^\lambda e^{rx}$ where λ is non negative integer, $r \in \mathbb{R}$, and C is constant.
2) $Cx^\lambda e^{\alpha x}\cos(\beta x)$ or $Cx^\lambda e^{\alpha x}\sin(\beta x)$,where α , $\beta \in \mathbb{R}$ such that $\beta \neq 0$.

For example

$$g(x) = 3x^2 - 2x + 5e^{3x} - xe^{2x}\sin x + 5\cos(2x) + x^2e^{-5x}.$$

On other hand , the functions

$$f(x) = \frac{1}{x} \; and \;\; f(x) = \ln(2x),$$

are not of these types. Substituting $y = e^{mx}$ in the equation (2) we obtain the characteristic equation

$$a_n m^n + a_{n-1} m^{n-1} + ... + a_1 m + a_0 = 0, \tag{5}$$

where a_0 ,a_1 , ..., a_n are constants. We have the following cases

Case (1) We suppose $g(x) = C\, x^{\lambda} e^{rx}$.

a) If r is not a root of (5), then the particular solution of (1) is the form

$$y_p(x) = (A_0 + A_1 x + ...A_{\lambda} x^{\lambda}) e^{rx}.$$

b) If r is a simple root of (5), then

$$y_p(x) = x(A_0 + A_1 x + ...A_{\lambda} x^{\lambda}) e^{rx}.$$

c) If r is a root of (5) with multiplicity k $(1 \le k \le n)$ then

$$y_p(x) = x^k (A_0 + A_1 x + ...A_{\lambda} x^{\lambda}) e^{rx}.$$

Case (2) We suppose that

$$g(x) = C x^{\lambda} e^{\alpha x} \cos(\beta x) \;\text{ or }\; g(x) = C x^{\lambda} e^{\alpha x} \sin(\beta x).$$

a) If $\alpha + i\,\beta$, where $\beta \neq 0$ is not root of (5), then the particular solution of (1) is the form

$$y_p(x) = (A_0 + A_1 x + ...A_{\lambda} x^{\lambda}) e^{\alpha x} \cos(\beta x) + (B_0 + B_1 x + ... + B_{\lambda} x^{\lambda}) e^{\alpha x} \sin(\beta x).$$

b) If $\alpha + i\,\beta$ is a simple root of (5), then

$$y_p(x) = x(A_0 + A_1 x + ...A_{\lambda} x^{\lambda}) e^{\alpha x} \cos(\beta x) + x(B_0 + B_1 x + ... + B_{\lambda} x^{\lambda}) e^{\alpha x} \sin(\beta x).$$

c) If $\alpha + i\,\beta$ is a root of (5) with multiplicity k ($k \le n$), then

$$y_p(x) = x^k (A_0 + A_1 x + ...A_{\lambda} x^{\lambda}) e^{\alpha x} \cos(\beta x) + x^k (B_0 + B_1 x + ... + B_{\lambda} x^{\lambda}) e^{\alpha x} \sin(\beta x).$$

Remark(4.5.2) Substituting

$$g(x) = f_1(x) + f_2(x) + ... + f_k(x),$$

in the equation (1), where f_i , $i = 1, ..., k$ are continuous functions on $I = (a, b)$. Let $y_{i,p}$ be a particular solution of the differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a(x)y' + a_0(x)y = f_i(x),$$

then

$$y_p(x) = y_{1,p} + y_{2,p} + ... + y_{k,p}.$$

is a particular solution of (1).

Example (1) Find the general solution of the differential equation :

$$y'' - y = -2x^2 + 5 + 2e^x. \tag{7}$$

Solution

1) First we have to find the general solution of the differential equation :

$$y'' - y = 0.$$

For , we have $m^2 - 1 = 0$, hence $m = \mp 1$ then

$$y_c = c_1 e^x + c_2 e^{-x}.$$

2) The form of the particular solution of

$$y'' - y = -2x^2 + 5,$$

is

$$y_{1,p} = Ax^2 + Bx + C,$$

and the form of the particular solution of

$$y'' - y = 2e^x,$$

is

$$y_{2,p} = Dxe^x,$$

because $r = 1$ is a simple root of the characteristic equation. Thus the particular solution of (7) is

$$y_p = y_{1,p} + y_{2,p} = Ax^2 + Bx + C + Dxe^x.$$

Now we have to find the constants A, B, C , and D by substituting y_p and y_p'' in differential equation (7) and we find

$$y_p'' - y_p = -Ax^2 - Bx + 2A - C + 2De^x = -2x^2 + 5 + 2e^x.$$

Equating coefficients of similar terms (because the functions $x^2, 1$ and are e^x linearly independent on $\mathbb{R}$), we obtain the following system of equation $A = 2$, $B = 0$, $2A - C = 5$, and $2D = 2$. Thus we have $A = 2$, $B = 0$, $C = -1$, and $D = 1$.Then the particular solution of (7) is

$$y_p = 2x^2 - 1 + xe^x,$$

and the general solution of the differential equation of (7) is

$$y = y_c + y_p = c_1e^x + c_2e^{-x} + 2x^2 - 1 + xe^x.$$

Example (2) Find only the form of particular solution of the differential equation.

$$y'' + 2y' - 3y = 3x^2e^x + e^{2x} + x\sin x + (2 + 3x). \tag{8}$$

Solution

1) the characteristic equation of

$$y'' + 2y' - 3y = 0,$$

is

$$m^2 + 2m - 3 = 0 = (m + 3)(m - 1) = 0. \tag{9}$$

Then we have two roots $m = -3$, $m = 1$.

2)

$$g(x) = 3x^2e^x + e^{2x} + x\sin x + (2 + 3x).$$

We observe that $r = 1$ is a root of (9) and $r = 2$, $r = 0$ and $m = \mp i$ are not roots of (9), then the particular solution of the differential equation (8) is

$$y_p = x(A_0+A_1x+A_2x^2)e^x+A_3e^{2x}+(A_4+A_5x)\sin x+(A_6+A_7x)\cos x+(A_8+A_9x).$$

Example (3) Find the general solution of the differential equation

$$y'' - 2y' + y = 2e^x - 3e^{-x}. \tag{10}$$

Solution

1) the characteristic equation of

$$y'' - 2y' + y = 0,$$

is

$$m^2 - 2m + 1 = (m-1)^2 = 0. \tag{11}$$

Then the general solution of the homogeneous differential equation is

$$y_c = c_1 e^x + c_2 x e^x.$$

2)

$$g(x) = 2e^x - 3e^{-x}.$$

We see that $r = 1$ is a root of (11) with multiplicity 2 and $r = -1$ is not root of (11) , then the particular solution of(11) is

$$y_p = Ax^2 e^x + Be^{-x},$$

hence we have

$$y_p'' - 2y'_p + y_p = 2Ae^x + 4Be^{-x} = 2e^x - 3e^{-x}.$$

Equating the coefficients of e^x and Equating the coefficients of e^{-x} , we obtain $A = 1$, $B = -\frac{3}{4}$, then

$$y_p = x^2 e^x - \frac{3}{4} e^{-x}.$$

Then the general solution of the differential equation of (10) is

$$y = y_c + y_p = c_1 e^x + c_2 x e^x + x^2 e^x - \frac{3}{4} e^{-x}.$$

Example (4) Determine the form of a particular solution of

$$a)\ y'' - 8y' + 25y = (5x^3 - 7)e^{-x}. \tag{12}$$

$$b)\ y'' + 4y = x\cos(2x). \tag{13}$$

Solution

a) The characteristic equation of

$$y'' - 8y' + 25y = 0,$$

is

$$m^2 \quad - 8m + 25 = 0,$$

which has the roots $m = 4 \mp 3i$.

Let

$$g(x) = (5x^3 - 7)e^{-x}.$$

But $r = -1$ is not root of the characteristic equation, then the particular solution of (12) is the form

$$y_p = (A_0 + A_1 x + A_2 x^2 + A_3 x^3)e^{-x}.$$

b) The characteristic equation of

$$y'' + 4y = 0,$$

is $m^2 + 4 = 0$, which has the roots $m = \mp 2i$.
Let

$$g(x) = x\cos(2x).$$

Since $m = 2i$ is a root of the characteristic equation, then the particular solution of (13) is the form

$$y_p = x(A_0 + A_1 x)\cos(2x) + x(B_0 + B_1 x)\sin(2x).$$

Example (5) Determine the form of a particular solution of

$$a)\ y'' - 6y' + 9y = 6x^2 + 2 - 12x^2 e^{3x}. \tag{14}$$

$$b) y^{(4)} + 2y'' + y = x\sin x + (x-1)e^{-x}. \tag{15}$$

Solution

a) The characteristic equation of

$$y'' - 6y' + 9y = 0,$$

is

$$m^2 - 6m + 9 = (m-3)^2 = 0. \tag{16}$$

Then $m = 3$ is a root of (16) with multiplicity 2. Corresponding to

$$f_1(x) = 6x^2 + 2,$$

we have

$$y_{1,p} = (A_0 + A_1 x + A_2 x^2).$$

Corresponding to

$$f_2(x) = -12x^2e^{3x},$$

we have

$$y_{2,p} = x^2(B_0 + B_1x + B_2x^2)e^{3x}.$$

Thus the particular solution of (14) is the form

$$y_p = y_{1,p} + y_{2,p} = (A_0 + A_1x + A_2x^2) + x^2(B_0 + B_1x + B_2x^2)e^{3x}.$$

b) The characteristic equation of

$$y^{(4)} + 2y'' + y = 0,$$

is

$$m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0. \tag{17}$$

Then $m = \mp i$ are the root of (17) with multiplicity 2, and $r = -1$ is not root of (17) , then corresponding to

$$f_1(x) = x \sin x \ ,$$

we have

$$y_{1,p} = x^2(A_0 + A_1x)\sin x + x^2(B_0 + B_1x)\cos x.$$

Now corresponding to

$$f_2(x) = (x-1)e^{-x},$$

we have

$$y_{2,p} = (C_0 + C_1x)e^{-x}.$$

Thus the particular solution of (15) is the form

$$y_p = y_{1,p} + y_{2,p} = x^2(A_0 + A_1x)\sin x + x^2(B_0 + B_1x)\cos x + (C_0 + C_1x)e^{-x}.$$

Example (6) Solve the differential equation

$$y''' + y'' = e^x \cos(x). \tag{18}$$

Solution

1) first we find the general solution of

$$y''' + y'' = 0. \tag{19}$$

The characteristic equation of (19) is

$$m^3 + m^2 = 0,$$

we have the roots $m = 0$ with multiplicity 2, and $m = -1$, then the the general solution of (19) is

$$y_c = c_1 + c_2 x + c_3 e^{-x}.$$

2) Now we have to find particular solution of (18). For, since $m = 1 + i$ is not a root of the characteristic equation,then we have

$$y_p = Ae^x \cos x + Be^x \sin x.$$

Then we obtain

$$y''' + y_p'' = (-2A + 4B)e^x \cos x + (-4A - 2B)e^x \sin x = e^x \cos x,$$

hence we get

$$-2A + 4B = 1 \ , \quad \text{and} \quad -4A - 2B = 0$$

then we obtain $A = -\frac{1}{10}$ and $B = \frac{1}{5}$, so the particular solution of (18) is

$$y_p = -\frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x.$$

So the general solution of (18) is

$$y = y_c + y_p = c_1 + c_2 x + c_3 e^{-x} + -\frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x.$$

Example (7) Find the general solution of the differential equation

$$y^{(4)} + y''' = 1 - e^{-x}. \tag{20}$$

Solution

1) the characteristic equation of

$$y^{(4)} + y''' = 0, \tag{21}$$

is

$$m^3(m + 1) = 0. \tag{22}$$

Then the roots of (22) are $m = 0$ with multiplicity 3 and $m = -1$, hence the general solution of (21) is

$$y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}.$$

2) Let

$$g(x) = 1 - e^{-x},$$

as $r = 0$ is a root of (22) with multiplicity 3 and $r = -1$ is a simple root of (22), then the particular solution of (20) is the form

$$y_p = Ax^3 + Bxe^{-x},$$

and we have

$$y_p^{(4)} + y_p''' = 6A - Be^{-x} = 1 - e^{-x}.$$

We get $A = \frac{1}{6}$ and $B = 1$. So that the particular solution of (20) is

$$y_p = \frac{1}{6}x^3 + xe^{-x}.$$

Then the general solution of the differential equation of (20) is

$$y = y_c + y_p = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} + \frac{1}{6}x^3 + xe^{-x}.$$

Example (8) Solve the differential equation

$$y'' + 4y = \sin(2x) + e^x. \tag{23}$$

Solution

1) the characteristic equation of

$$y'' + 4y = 0, \tag{24}$$

is

$$m^2 + 4 = 0. \tag{25}$$

Then the root of (25) are $m = \mp 2i$, hence

$$y_c = c_1 \sin(2x) + c_2 \cos(2x).$$

2) Let

$$g(x) = \sin(2x) + e^x.$$

As $m = 2i$ is a simple root of (25) and $r = 1$ is not a root of (25), then

$$y_p = Ax\cos(2x) + Bx\sin(2x) + Ce^x.$$

We get

$$y_p'' + 4y_p = -4A\sin(2x) + 4B\cos(2x) + 5Ce^x = \sin(2x) + e^x.$$

By identifying the coefficients in two sides we obtain $A = -\frac{1}{4}$, $B = 0$, and $C = \frac{1}{5}$.

So the particular solution of (23) is

$$y_p = -\frac{1}{4}x\cos(2x) + \frac{1}{5}e^x,$$

and the general solution of the differential equation of (23) is

$$y = y_c + y_p = c_1\sin(2x) + c_2\cos(2x) - \frac{1}{4}x\cos(2x) + \frac{1}{5}e^x.$$

Example(9) Solve the initial value problem (IVP)

$$\begin{cases} y'' + y = \cos x - \sin(2x) \\ y(\frac{\pi}{2}) = 0 \ , \quad y'(\frac{\pi}{2}) = 0. \end{cases} \tag{26}$$

Solution

1) the characteristic equation of

$$y'' + y = 0,$$

is $m^2 + 1 = 0$, hence we have two roots $m = \mp i$ and the general solution of the differential equation of the homogeneous differential equation is

$$y_c = c_1\cos x + c_2\sin x.$$

2) Let

$$g(x) = \cos x - \sin(2x),$$

since $m = i$ is a simple root of $m^2 + 1 = 0$ then

$$y_p = Ax\cos x + Bx\sin x + C\cos(2x) + D\sin(2x).$$

Then

$$y_p'' + y_p = -2A\sin x + 2B\cos x - 3\cos(2x) - 3D\sin(2x) = \cos x - \sin(2x).$$

By identifying the coefficients in two sides we obtain $A = 0$, $B = \frac{1}{2}$, $C = 0$, and $D = \frac{1}{3}$. Thus

$$y_p = \frac{1}{2}x\sin x + \frac{1}{3}\sin(2x),$$

and the general solution of the differential equation of (26) is

$$y = y_c + y_p = c_1\cos x + c_2\sin x + \frac{1}{2}x\sin x + \frac{1}{3}\sin(2x).$$

3) From the initial conditions we get

$$y(\frac{\pi}{2}) = c_2 + \frac{\pi}{4} = 0\ ,$$

and

$$y'(\frac{\pi}{2}) = -c_1 + \frac{1}{2} + \frac{2}{3} = 0,$$

hence $c_1 = \frac{7}{6}$ and $c_2 = -\frac{\pi}{4}$. So the the solution of the of the IVP (27) is

$$y = \frac{7}{6}\cos x - \frac{\pi}{4}\sin x + \frac{1}{2}x\sin x + \frac{1}{3}\sin(2x).$$

Example(10) Solve the initial value problem (IVP)

$$\begin{cases} y''' + 8y = 2x - 5 + 8e^{-2x} \\ y(0) = -5 \quad , \ y'(0) = 3 \quad , \quad y''(0) = -4. \end{cases} \qquad (27)$$

Solution

1) the characteristic equation of

$$y''' + 8y = 0,$$

is

$$m^3 + 8 = (m+2)(m^2 - 2m + 4) = 0, \qquad (28)$$

hence

$$m = -2\ , m = 1 \mp \sqrt{3}i\ ,$$

and the general solution of the homogeneous equation is

$$y_c = c_1e^{-2x} + c_2e^{x}\cos(\sqrt{3}x) + c_3e^{x}\sin(\sqrt{3}x).$$

2) Let

$$g(x) = 2x - 5 + 8e^{-2x}.$$

We note that $r = 0$ is not a root of (28) and $r = -2$ is a simple root, then

$$y_p = Ax + B + Cxe^{-2x}.$$

So we have·

$$y_p''' + 8y_p = 12Ce^{-2x} + 8Ax + 8B = 2x - 5 + 8e^{-2x}.$$

By identifying the coefficients in two sides, we get $A = \frac{1}{4}$, $B = \frac{-5}{8}$ and $C = \frac{2}{3}$, hence the general solution of (28) is

$$y = y_c + y_p = c_1 e^{-2x} + c_2 e^x \cos(\sqrt{3}x) + c_3 e^x \sin(\sqrt{3}x) + \frac{x}{4} - \frac{5}{8} + \frac{2}{3}xe^{-2x}.$$

From the initial conditions, we get

$$y(0) = c_1 + c_2 - \frac{5}{8} = -5, \; y'(0) = -2c_1 + c_2 + \sqrt{3}c_3 + \frac{1}{4} + \frac{2}{3} = 3,$$

and

$$y''(0) = 4c_1 - 2c_2 + 2\sqrt{3}c_3 - \frac{8}{3} = -4.$$

The solution of this system is

$$c_1 = -\frac{23}{12}, c_2 = -\frac{59}{24}, \; c_3 = \frac{17\sqrt{3}}{72}.$$

Then the solution of the IVP is

$$y = -\frac{23}{12}e^{-2x} - \frac{59}{24}e^x \cos(\sqrt{3}x) + \frac{17\sqrt{3}}{72}e^x \sin(\sqrt{3}x) + \frac{x}{4} - \frac{5}{8} + \frac{2}{3}xe^{-2x}.$$

Example(11) In application the input function $g(x)$ is often discontinuous. Solve the initial value problem

$$\begin{cases} y'' + 4y = g(x) \\ y(0) = 1, \; y'(0) = 2. \end{cases} \tag{29}$$

$$g(x) = \begin{cases} \sin x & ; \quad 0 \le x \le \frac{\pi}{2} \\ 0 & ; \quad x > \frac{\pi}{2}. \end{cases} \tag{30}$$

Solution First we solve the problem on the two intervals and then we find the solution so that y and y' are continuous at $x = \frac{\pi}{2}$.

1) The characteristic equation of

$$y'' + 4y = 0 \ ,$$

is $m^2 + 4 = 0$,then we have two roots $m = \mp 2i$. Hence

$$y_c = c_1 \cos(2x) + c_2 \sin(2x).$$

Now if $g(x) = \sin x$ for $0 \leq x \leq \frac{\pi}{2}$, then g is continuous on $\left[0 \ , \ \frac{\pi}{2}\right]$, so

$$y_p = A \sin x + B \cos x,$$

hence

$$y_p'' + 4y_p = 3A \sin x + 3B \cos x = \sin x,$$

which implies that $A = \frac{1}{3}$. $B = 0$. So the general solution of

$$y'' + 4y = \sin x \ ; \quad 0 \leq x \leq \frac{\pi}{2},$$

is

$$y = y_c + y_p = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{3} \sin x.$$

2) But

$$y'(x) = -2c_1 \sin(2x) + 2c_2 \cos(2x) + \frac{1}{3} \cos x,$$

and from the conditions (29) we have

$$y(0) = c_1 = 1 \text{ and } y'(0) = 2c_2 + \frac{1}{3} = 2,$$

hence $c_1 = 1$ and $c_2 = \frac{5}{6}$. Thus the solution of the IVP

$$y'' + 4y = \sin x, \ y(0) = 1, \ y'(0) = 2 \ ; \quad 0 \leq x \leq \frac{\pi}{2},$$

is

$$y = \cos(2x) + \frac{5}{6} \sin(2x) + \frac{1}{3} \sin x. \tag{31}$$

3) The general solution of

$$y'' + 4y = 0 \ ; x > \frac{\pi}{2}, \tag{32}$$

So the function

$$y = c_3 \cos(2x) + c_4 \sin(2x). \qquad (33)$$

$$y = f(x) = \begin{cases} \cos(2x) + \frac{5}{6}\sin(2x) + \frac{1}{3}\sin x & ; \ 0 \leq x \leq \frac{\pi}{2} \\ c_3\cos(2x) + c_4 \sin(2x) & ; \ x > \frac{\pi}{2} \end{cases}$$

is a solution of the IVP (29)) and (32) such that y and y' are continuous at $x = \frac{\pi}{2}$. For this we should have

$$\lim_{x \to (\frac{\pi}{2})^+} y(x) = \lim_{x \to (\frac{\pi}{2})^-} y(x) \ y(\frac{\pi}{2}),$$

which implies

$$-1 + \frac{1}{3} = -c_3 \ or \ c_3 = \frac{2}{3}.$$

But

$$y'(x) = f'(x) = \begin{cases} -2\sin(2x) + \frac{5}{3}\cos(2x) + \frac{1}{3}\cos x & ; \ 0 \leq x \leq \frac{\pi}{2} \\ -2c_3\sin(2x) + 2c_4\cos(2x) & ; \ x > \frac{\pi}{2}, \end{cases}$$

we should have

$$\lim_{x \to (\frac{\pi}{2})^+} y'(x) = \lim_{x \to (\frac{\pi}{2})^-} y'(x) = y'(\frac{\pi}{2}),$$

which implies

$$-2c_4 = -\frac{5}{3} \ or \ c_4 = \frac{5}{6}.$$

Finally the solution of the IVP (29), (30) and (32) is

$$y = f(x) = \begin{cases} \cos(2x) + \frac{5}{6}\sin(2x) + \frac{1}{3}\sin x & ; \ 0 \leq x \leq \frac{\pi}{2} \\ \frac{2}{3}\cos(2x) + \frac{5}{6}\sin(2x) & ; \ x > \frac{\pi}{2}. \end{cases}$$

Exercises (4.5)

In problems 1 through 20 find the general solution of the given differential equations by using undetermined coefficients method .

1) $y'' + 3y' + 2y = 5.$
2) $y'' - 10y' + 25y = 30x + 3.$
3) $y'' + y' - 6y = 2x.$
4) $\frac{1}{4}y'' + y' + y = x^2 - 2x.$
5) $y'' - 8y' + 20y = 3x^2 - 4xe^x.$

6) $4y'' - 4y' - 3y = \cos(2x)$.
7 $y'' + 3y' = -48x^2e^{3x}$.
8) $y'' + 2y' = 2x + 5 - e^{-2x}$.
9) $y'' - y' - 2y = 2xe^{-x} + x^2$.
10) $y'' - y = 4\cosh(x) = 2(e^x + e^{-x})$.
11) $y'' - 7y' - 8y = e^x(x^2 + 2)$.
12) $y'' - 5y' + 4y = e^{2x}(\cos x + \sin x)$.
13) $y''' - 3y'' + 3y' - y = x^2 + 5e^x$.
14) $y''' + y = x + xe^x$.
15) $y''' - 6y'' = 3 - \cos x$.
16) $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$.
17) $y''' - 3y'' + 3y' - y = x - 4e^x$.
18) $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$.
19) $y^{(4)} + 2y'' + y = (x-1)^2$.
20) $y^{(4)} - y'' = 4x + 2xe^{-x}$.

In problems 21 through 22 use trigonometric identity as an aid in finding a particular solution of the given differential equation.

21) $y'' + 4y = 8\sin^2(x)$.
22) $y'' + y = \sin x \cos(2x)$.

In problems 23 through 30 find only the form of the particular solution of the given differential equation by using undetermined coefficients method.

23) $y'' - y = e^x + \sin x$.
24) $y'' - 4y' + 4y = e^{2x}$.
25) $y''' + y' = x^2 - 3x + 1$.
26) $y'' - y = x^2e^x$.
27) $y^{(6)} - 3y^{(3)} = 3x + 1$.
28) $y''' - y' = x^5 + \cos x$.
29) $y''' + 3y'' - 4y = e^{-2x}$.
30) $y'' + 4y = 4x^3 - 8x^2 - 14x + 7$.

In exercises 31 through 35 answer true or false.

31) A particular solution of $y'' + 3y' + 2y = e^x$ is the form Axe^x.
32) A particular solution of $y'' - 3y' + 2y = e^x$ is the form Axe^x.
33) A particular solution of $y'' + y = \frac{1}{x}$ can not be found by the method of undetermined coefficients .
34) A particular solution of $y'' + y = \cos x$ is the form $Ax\cos x + Bx\sin x$.
35) A particular solution of $y'' - 3y = x\ln x$ can be found by the method of undetermined coefficients .

In exercises 36 through 50 solve the initial value problems (IVP).

36) $\begin{cases} y'' + y = -2 \\ y(\frac{\pi}{4}) = \frac{1}{2} \;,\; y'(\frac{\pi}{4}) = -1. \end{cases}$

37) $\begin{cases} 2y'' + 3y' - 2y = 14x^2 - 4x - 11 \\ y(0) = y'(0) = 0. \end{cases}$

38) $\begin{cases} 5y'' + y' = -6x \\ y(0) = 0,\; y'(0) = -10. \end{cases}$

39) $\begin{cases} y'' + 4y' + 4y = (3+x)e^{-2x} \\ y(0) = 2 \;, y'(0) = 5. \end{cases}$

40) $\begin{cases} y'' + 4y' + 5y = 35e^{-5x} \\ y(0) = -3 \;,\; y'(0) = 1. \end{cases}$

41) $\begin{cases} y'' - y = \frac{1}{2}(e^x + e^{-x}) \\ y(0) = 2 \;, y'(0) = 12. \end{cases}$

42) $\begin{cases} y'' + y = \cos x - \sin(2x) \\ y(\frac{\pi}{2}) = 0 \;, y'(\frac{\pi}{2}) = 0. \end{cases}$

43) $\begin{cases} y'' - 2y' - 3y = 2\cos^2(x) \\ y(0) = -\frac{1}{3} \;,\; y'(0) = 0. \end{cases}$

44) $\begin{cases} y''' + y' = x \\ y(0) = 0 \;,\; y'(0) = 1 \;\; y''(0) = 0 \end{cases}$

45) $\begin{cases} y''' - 2y'' + y' = 2 - 24e^x + 40e^{5x} \\ y(0) = \frac{1}{2} \;,\; y'(0) = \frac{5}{2} \;\; y''(0) = -\frac{9}{2} \end{cases}$

46) $\begin{cases} y^{(4)} - y''' = x + e^x \\ y(0) = \;\; y'(0) = 0 \;,\;\; y''(0) = y'''(0) = 0 \end{cases}$

47) $\begin{cases} y^{(4)} - y = x^2 + 1 \\ y(0) = \;\; y'(0) = 1 \;\;,\; y''(0) = y'''(0) = 0. \end{cases}$

48) $\begin{cases} y''' - 3y'' + 3y' - 1 = 0 \\ y(0) = 1 \;,\; y'(0) = -1 \;, y''(0) = 0. \end{cases}$

49) $\begin{cases} y^{(5)} - 6y^{(4)} + 9y''' = x \\ y(0) = \;\; y'(0) = 0,\; y''(0) = y'''(0) = 1,\; y^{(4)}(0) = -1. \end{cases}$

50) $\begin{cases} y^{(4)} + 2y''' + 3y'' + 2y' + 1 = x^2 + 1 \\ y(0) = \;\; y'(0) = -1,\;\; y''(0) = y'''(0) = 0. \end{cases}$

4.6 Variation of parameters

The method of variation of parameter, like the method of undetermined coefficients, is used to compute a particular solution of nonhomogeneous differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots\dots + a_1(x)y' + a_0(x)y = g(x). \qquad (1)$$

The undetermined coefficients, however requires that the coefficients a_0 , a_1 , ..., and a_n of the differential equation be constants and the function g must be one of the types described in the section (4.5.1.). When we encounter differential equation for which we cannot apply the method of undetermined coefficients, we use the variation of parameters method which is better than undetermined coefficients method and more efficient, where we can apply the variation of parameters even the coefficients a_0 , a_1 , ..., and a_n are functions.

Now we suppose that we know the general solution

$$y_c = c_1y_1 + c_2y_2 + \dots\dots + c_n\, y_n\ . \qquad (2)$$

of the homogeneous differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots\dots + a_1(x)y' + a_0(x) = 0. \qquad (3)$$

Where $a_1(x)$, $a_2(x)$,......, $a_n(x)$ are continuous functions defined on an interval I such that $a_n(x) \neq 0$ for all $x \in I$.

We have the following theorem which will be proved in the case $n = 2$.

Theorem(4.6.1) If y_1 , y_2 , ..., y_n form a fundamental set of solution for the equation (3) and if u_1 , u_2 , ..., u_n satisfy the system of equations

$$\begin{cases} y_1u_1' + y_2u_2' + \dots\dots + y_nu_n' = 0 \\ y_1'u_1' + y_2'u_2' + \dots\dots y_n'u_n' = 0 \\ \dots\dots\dots\dots\dots\dots \\ y_1^{(n-2)}u_1' + y_2^{(n-2)}u_2' + \dots\dots + y_n^{(n-2)}u_n' = 0 \\ y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \dots\dots + y_n^{(n-1)}u_n' = \frac{g(x)}{a_n(x)}. \end{cases} \qquad (4)$$

Then

$$y_p = y_1u_1 + y_2u_2 + \dots\dots + y_nu_n$$

is a particular solution of the differential equation (1).

Proof **(special case $n = 2$)**

By hypothesis y_1 and y_2 form a fundamental set of solutions for differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{5}$$

and that u_1 and u_2 satisfy the system of algebraic equations

$$\begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = \frac{g(x)}{a_2(x)}. \end{cases} \tag{6}$$

Then the particular solution of the

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x),$$

is

$$y_p = y_1u_1 + y_2u_2.$$

Here we suppose that a_0 ,a_1 ,a_2 and g are continuous functions on an interval I such that $a_2(x) \neq 0$ for all $x \in I$. But

$$W(x, y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \neq 0,$$

for all $x \in I$, because y_1 and y_2 are solutions of (5) and linearly independent on I .The the system (6) has a unique solution u_1' , u_2'. From u_1' and u_2' we deduce u_1 and u_2 via integration . All that remains to shown that the function

$$y_p = y_1u_1 + y_2u_2, \tag{7}$$

is a solution of the differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x). \tag{8}$$

Indeed, from (7) we have

$$\begin{aligned} y_p' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2', \\ &= (y_1u_1' + y_2u_2') + (u_1y_1' + u_2y_2') = u_1y_1' + u_2y_2', \end{aligned}$$

$$y_p'' = (u_1'y_1' + u_2'y_2') + (u_1y_1'' + u_2y_2'') = \frac{g(x)}{a_2(x)} + (u_1y_1'' + u_2y_2''),$$

$$
\begin{aligned}
& a_2(x)y_p'' + a_1(x)y'_p + a_0(x)y_p \\
= \; & a_2(x)\left[\frac{g(x)}{a_2(x)} + (u_1y_1'' + u_2y_2'')\right] + a_1(x)\left[u_1y_1' + u_2y_2'\right] + a_0(x)\left[y_1u_1 + y_2u_2\right], \\
= \; & u_1\left[a_2(x)y_1'' + a_1(x)y'_1 + a_0(x)y_1\right] + u_2\left[a_2(x)y_2'' + a_1(x)y'_2 + a_0(x)y_2\right] + g(x), \\
= \; & g(x).
\end{aligned}
$$

Therefore, y_p satisfies the equation (1) for $n = 2$ and the proof is complete. The proof in general case follows exactly in the same manner. We note that the variation of parameter method applies to any nonhomogeneous differential equation no matter what the coefficients and the function g happen to be. The reason for introducing the method of undetermined coefficients that it is sometimes quicker and easier to apply.

Example(1) Solve the differential equation

$$y'' + y = \csc x \quad ; \quad 0 < x < \pi. \tag{9}$$

Solution

1) The general solution of

$$y'' + y = 0,$$

is

$$y_c = c_1 \sin x + c_2 \cos x.$$

2) The particular solution of

$$y'' + y = \csc x,$$

is the form

$$y_p = y_1u_1 + y_2u_2,$$

where

$$y_1 = \sin x \quad and \quad y_2 = \cos x.$$

The functions u_1 and u_2 are determined from the system of equations

$$
\begin{cases}
(\sin x)u_1' + (\cos x)u_2' = 0 \\
(\cos x)u_1' - (\sin x)u_2' = \csc x.
\end{cases}
$$

But

$$W(x, y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1,$$

$$W_1 = \begin{vmatrix} 0 & \cos x \\ \csc x & -\sin x \end{vmatrix} = -\cot x,$$

$$W_2 = \begin{vmatrix} \sin x & 0 \\ \cos x & \csc x \end{vmatrix} = 1,$$

Hence

$$u_1' = \frac{W_1}{W} = \cot x,$$

then

$$u_1 = \ln(\sin x).$$

But

$$u_2' = -1,$$

hence $u_2 = -x$. Therefore we have

$$y_p = y_1 u_1 + y_2 u_2 = \sin x. \ln(\sin x) - x \cos x,$$

and the general solution of (9) is

$$y = y_c + y_p = c_1 \sin x + c_2 \cos x + \sin x. \ln(\sin x) - x \cos x.$$

Example(2) Solve the differential equation

$$y'' - 4y' + 4y = (x + 1)e^{2x}. \tag{10}$$

Solution

1) The general solution of

$$y'' - 4y' + 4y = 0,$$

is

$$y_c = c_1 e^{2x} + c_2 x e^{2x}.$$

2) Let

$$y_1 = e^{2x} \;\; and \;\; y_2 = x e^{2x}.$$

So we have

$$W(x, y_1, y_2) = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix} = e^{4x},$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = -x(x+1)e^{4x},$$

and

$$W_2(x, y_1, y_2) = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x},$$

hence

$$u_1' = \frac{W_1}{W} = -x(x+1) = -x^2 - x,$$

so

$$u_1 = -\frac{x^3}{3} - \frac{x^2}{2}$$

But

$$u_2' = \frac{W_2}{W} = x + 1,$$

then

$$u_2 = \frac{x^2}{2} + x.$$

Therefore

$$y_p = y_1 u_1 + y_2 u_2 = (-\frac{x^3}{3} - \frac{x^2}{2})e^{2x} + x(\frac{x^2}{2} + x)e^{2x} = (\frac{x^3}{6} + \frac{x^2}{2})e^{2x},$$

and The general solution of (10) is

$$y = y_c + y_p = c_1 e^{2x} + c_2 x e^{2x} + (\frac{x^3}{6} + \frac{x^2}{2})e^{2x}.$$

In this example we can use the undetermined coefficients, where

$$y_p = x^2(A + Bx)e^{2x}.$$

Example(3) Solve the differential equation

$$y'' - y = \frac{1}{x} \quad ; \quad x > 0. \tag{11}$$

Solution
1) The general solution of

$$y'' - y = 0,$$

is

$$y_c = c_1 e^x + c_2 e^{-x}.$$

2) Let

$$y_1 = e^x \text{ and } y_2 = e^{-x}.$$

Then we have

$$W(x, y_1, y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2,$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & e^{-x} \\ \frac{1}{x} & -e^{-x} \end{vmatrix} = -\frac{1}{x}e^{-x},$$

$$W_2(x, y_1, y_2) = \begin{vmatrix} e^x & 0 \\ e^x & \frac{1}{x} \end{vmatrix} = \frac{1}{x}e^x,$$

hence

$$u_1' = \frac{W_1}{W} = \frac{1}{2}\frac{1}{x}e^{-x},$$

and

$$u_1(x) = \frac{1}{2}\int_{x_0}^{x} \frac{e^{-t}}{t}dt \,.$$

Where x_0 and $x \in (0, \infty)$.

But

$$u_2' = \frac{W_2}{W} = -\frac{1}{2}\frac{1}{x}e^x,$$

so

$$u_2 = -\frac{1}{2}\int_{x_0}^{x} \frac{e^t}{t}dt \,.$$

Then

$$y_p = y_1 u_1 + y_2 u_2 = \frac{1}{2}e^x \int_{x_0}^{x} \frac{e^{-t}}{t}dt - \frac{1}{2}e^{-x}\int_{x_0}^{x} \frac{e^t}{t}dt \,,$$

and the general solution of (11) is

$$y = y_c + y_p = c_1 e^x + c_2 e^{-x}. + \frac{1}{2}e^x \int_{x_0}^{x} \frac{e^{-t}}{t}dt - \frac{1}{2}e^{-x}\int_{x_0}^{x} \frac{e^t}{t}dt.$$

Example(4) Solve the Differential equation

$$y'' - 3y' + 2y = \frac{1}{1 + e^{-x}}. \tag{12}$$

Solution

1) The general solution of

$$y'' - 3y' + 2y = 0.$$

is

$$y_c = c_1 e^x + c_2 e^{2x}.$$

2) Let

$$y_1 = e^x \text{ and } y_2 = e^{2x},$$

then

$$W(x, y_1, y_2) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x},$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & e^{2x} \\ \frac{1}{1+e^{-x}} & 2e^{2x} \end{vmatrix} = \frac{-e^{2x}}{1+e^{-x}},$$

$$W_2(x, y_1, y_2) = \begin{vmatrix} e^x & 0 \\ e^x & \frac{1}{1+e^{-x}} \end{vmatrix} = \frac{e^x}{1+e^{-x}},$$

hence

$$u_1' = \frac{W_1}{W} = -\frac{e^{-x}}{1+e^{-x}}$$

and

$$u_1(x) = -\int \frac{e^{-x}}{1+e^{-x}} dx = \ln(1+e^{-x}).$$

But

$$u_2' = \frac{W_2}{W} = \frac{e^{-2x}}{1+e^{-x}},$$

and

$$u_2 = \int \frac{e^{-2x}}{1+e^{-x}} dx = -(1+e^{-x}) + \ln(1+e^{-x}),$$

so we have

$$\begin{aligned} y &= y_c + y_p = (c_1 - 1)e^x + (c_2 - 1)e^{2x} + (e^x + e^{2x})\ln(1+e^{-x}), \\ &= c_3 e^x + c_4 e^{2x} + (e^x + e^{2x})\ln(1+e^{-x}), \end{aligned}$$

where $c_3 = c_1 - 1$ and $c_4 = c_2 - 1$.

Remark(4.6.2) When computing the indefinite integral of u_1' and u_2', we need not introduce any constant .Because

$$\begin{aligned} y &= y_c + y_p = c_1y_1 + c_2y_2 + (u_1+a)y_1 + (u_2+b)y_2 \\ &= (c_1+a)y_1 + (c_2+b)y_2 + u_1y_1 + u_2y_2 = c_3y_1 + c_4y_2 + u_1y_1 + u_2y_2, \end{aligned}$$

where $c_3 = c_1 + a$ and $c_4 = c_2 + b$.

Example(5) Solve the Differential equation

$$4y'' - 4y' + y = e^{\frac{x}{2}}\sqrt{1-x^2} \quad ; \quad |x| \le 1. \tag{13}$$

Solution

1) The general solution of

$$4y'' - 4y' + y = 0,$$

is

$$y_c = c_1e^{\frac{x}{2}} + c_2xe^{\frac{x}{2}}.$$

2) Let

$$y_1 = e^{\frac{x}{2}} \text{ and } y_2 = xe^{\frac{x}{2}},$$

then

$$W(x, y_1, y_2) = \begin{vmatrix} e^{\frac{x}{2}} & xe^{\frac{x}{2}} \\ \frac{1}{2}e^{\frac{x}{2}} & e^{\frac{x}{2}} + \frac{x}{2}e^{\frac{x}{2}} \end{vmatrix} = e^x,$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & xe^{\frac{x}{2}} \\ \frac{1}{4}e^{\frac{x}{2}}\sqrt{1-x^2} & e^{\frac{x}{2}} + \frac{x}{2}e^{\frac{x}{2}} \end{vmatrix} = -\frac{1}{4}xe^x\sqrt{1-x^2},$$

$$W_2(x, y_1, y_2) = \begin{vmatrix} e^{\frac{x}{2}} & 0 \\ \frac{1}{2}e^{\frac{x}{2}} & \frac{1}{4}e^{\frac{x}{2}}\sqrt{1-x^2} \end{vmatrix} = \frac{1}{4}e^x\sqrt{1-x^2},$$

hence

$$u_1' = \frac{W_1}{W} = -\frac{1}{4}x\sqrt{1-x^2},$$

so

$$u_1 = -\frac{1}{4}\int x\sqrt{1-x^2}dx = \frac{1}{12}(1-x^2)^{\frac{3}{2}}.$$

But

$$u_2' = \frac{W_2}{W} = \frac{1}{4}\sqrt{1-x^2},$$

then

$$u_2 = \frac{1}{4}\int \sqrt{1-x^2}dx \ .$$

To evaluate this integral we put $x = \sin\theta$; $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and we find

$$u_2 = \frac{1}{8}\sin^{-1}(x) + \frac{1}{8}x\sqrt{1-x^2}.$$

So we have

$$y_p = y_1u_1 + y_2u_2 = \frac{1}{12}e^{\frac{x}{2}}(1-x^2)^{\frac{3}{2}} + \frac{x}{8}e^{\frac{x}{2}}\sin^{-1}(x) + \frac{1}{8}x^2e^{\frac{x}{2}}\sqrt{1-x^2}.$$

The general solution of (13) is

$$y = y_c + y_p = c_1e^{\frac{x}{2}} + c_2xe^{\frac{x}{2}} + \frac{1}{12}e^{\frac{x}{2}}(1-x^2)^{\frac{3}{2}} + \frac{x}{8}e^{\frac{x}{2}}\sin^{-1}(x) + \frac{1}{8}x^2e^{\frac{x}{2}}\sqrt{1-x^2}.$$

Example(6)

a) Use undetermined coefficients to find a particular solution of the Differential equation

$$y'' + 2y' + y = 4x^2 - 3. \qquad (14)$$

b)Use variation of parameters to find a particular solution of

$$y'' + 2y' + y = \frac{e^{-x}}{x} \quad ; \; x \neq 0\,. \qquad (15)$$

c) Find the general solution of the Differential equation

$$y'' + 2y' + y = 4x^2 - 3 + \frac{e^{-x}}{x}. \qquad (16)$$

Solution

a) The general solution of the Differential equation

$$y'' + 2y' + y = 0,$$

is

$$y_c = c_1e^{-x} + c_2xe^{-x}.$$

The particular solution of (14) is the form

$$y_p = Ax^2 + Bx + C\,,$$

then

$$y_p'' + 2y_p' + y_p = Ax^2 + (4A+B)x + (2A+2B+C) = 4x^2 - 3,$$

we deduce $A = 4$, $B = -16$ and $C = 21$, hence

$$y_p = 4x^2 - 16x + 21.$$

b) Let

$$y_1 = e^{-x} \quad \text{and} \quad y_2 = xe^{-x}.$$

Then

$$W(x, y_1, y_2) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x},$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & xe^{-x} \\ \frac{e^{-x}}{x} & e^{-x} - xe^{-x} \end{vmatrix} = -e^{-2x},$$

$$W_2(x, y_1, y_2) = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{e^{-x}}{x} \end{vmatrix} = \frac{e^{-2x}}{x},$$

hence

$$u_1' = \frac{W_1}{W} = -1,$$

and

$$u_1 = -x.$$

But

$$u_2' = \frac{W_2}{W} = \frac{1}{x},$$

then

$$u_2 = \ln|x|.$$

So the particular solution of (15) is

$$y_p = y_1u_1 + y_2u_2 = -xe^{-x} + xe^{-x}\ln|x|.$$

c) From a) and b) we deduce that the particular solution of (16) is

$$y_p = 4x^2 - 16x + 21 - xe^{-x} + xe^{-x}\ln|x|,$$

and the general solution of (16) is

$$y = y_c + y_p = c_1e^{-x} + c_2xe^{-x} + 4x^2 - 16x + 21 + -xe^{-x} + xe^{-x}\ln|x|.$$

Example (7) Find the general solution of the differential equation

$$x^2y'' - xy' + y = 4x\ln x \quad ; \quad x > 0. \tag{17}$$

Solution The equation (17) is *Euler's* nonhomogeneous differential equation, we can use three methods for solving this equation.

Method (1) The general solution of

$$x^2y'' - xy' + y = 0,$$

is

$$y_c = c_1x + c_2x\ln x.$$

Let

$$y_1 = x \text{ and } y_2 = x\ln x .$$

Then

$$y_p = y_1u_1 + y_2u_2,$$

where u_1 and u_2 satisfy the system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \tfrac{4x\ln x}{x^2} = \tfrac{4}{x}\ln x . \end{aligned}$$

Then

$$\begin{cases} u_1'x + u_2'x\ln x = 0 \\ u_1' + u_2'(1+\ln x) = \frac{4}{x}\ln x . \end{cases}$$

$$W(x, y_1, y_2) = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x \neq 0,$$

for all $x > 0$.

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & x\ln x \\ \frac{4}{x}\ln x & 1+\ln x \end{vmatrix} = -4(\ln x)^2,$$

$$W_2(x, y_1, y_2) = \begin{vmatrix} x & 0 \\ 1 & \frac{4}{x}\ln x \end{vmatrix} = 4\ln x,$$

then

$$u_1' = \frac{W_1}{W} = -\frac{4}{x}(\ln x)^2,$$

and

$$u_1 = -\frac{4}{3}(\ln x)^3.$$

But

$$u_2' = \frac{W_2}{W} = \frac{4}{x}\ln x,$$

hence

$$u_2 = 2(\ln x)^2,$$

$$y_p = y_1u_1 + y_2u_2 = -\frac{4}{3}x(\ln x)^3 + 2x(\ln x)^3 = \frac{2}{3}x(\ln x)^3,$$

and the general solution of (17) is

$$y = y_c + y_p = c_1x + c_2x\ln x + \frac{2}{3}x(\ln x)^3.$$

Method (2) (Reduction of order)
We know that $y_1 = x$ is a particular solution of

$$x^2y'' - xy' + y = 0,$$

then the solution of (17) is the form

$$y = y_1u = xu,$$

where u is a function will be determined by the following method. We have

$$y' = u + xu' \,, y'' = 2u' + xu'',$$

and

$$\left\{\begin{array}{c} x^2y'' - xy' + y = 2u'x^2 + x^3u'' - xu - x^2u' + xu = 4x\ln x \\ x^3u'' + x^2u' = 4x\ln x \\ u'' + \frac{1}{x}u' = \frac{4}{x^2}\ln x \end{array}\right.$$

Let $u' = z$, then

$$u'' = z',$$

and

$$z' + \frac{1}{x}z = \frac{4}{x^2}\ln x.$$

So we have a linear differential equation of order one which has the general solution

$$u' = z = \frac{c_1}{x} + \frac{2}{x}(\ln x)^2,$$

hence

$$u = \frac{2}{3}(\ln x)^3 + c_1(\ln x) + c_2.$$

So the general solution of (17) is

$$y = xu = c_1 x(\ln x) + c_2 x + \frac{2}{3}x(\ln x)^3.$$

Method (3) (using some transformation) Substituting $x = e^t$, then

$$xy'(x) = y'(t)\ ,\ x^2y''(x) = y''(t) - y'(t).$$

We have

$$x^2y'' - xy' + y = y''(t) - 2y'(t) + y = 4te^t. \tag{18}$$

The general solution of

$$y''(t) - 2y'(t) + y = 0,$$

is

$$y_c = c_1e^t + c_2te^t,$$

and the particular solution of (18) is the form

$$y_p = t^2(At + B)e^t = (At^3 + Bt^2)e^t,$$

hence

$$y_p''(t) - 2y_p'(t) + y_p = (6At + 2B)e^t = 4te^t,$$

then $A = \frac{2}{3}$ and $B = 0$.So the general solution of (18) is

$$y = y_c + y_p = c_1e^t + c_2te^t + \frac{2}{3}t^3e^t.$$

and the general solution of (17) is

$$y = c_1x + c_2x\ln x + \frac{2}{3}x(\ln x)^3, \text{ because } x = e^t \text{ and } t = \ln x.$$

Example (8) Find the general solution of the differential equation

$$5x^2y'' - 3xy' + 3y = \sqrt{x}\quad ;\quad x > 0. \tag{19}$$

Solution

1) We have to find the general solution of

$$5x^2y'' - 3xy' + 3y = 0. \tag{20}$$

By using the substituting $y = x^m$, we have

$$5m^2 - 8m + 3 = (5m-3)(m-1) = 0,$$

so we have two roots $m = \frac{3}{5}$ and $m = 1$. Then the general solution of (20) is

$$y_c = c_1x + c_2x^{\frac{3}{5}}.$$

2) Let $y_1 = x$ and $y_2 = x^{\frac{3}{5}}$, we have

$$y_p = y_1u_1 + y_2u_2,$$

where u_1 and u_2 are two functions satisfying the system

$$\begin{cases} u_1'x + u_2'x^{\frac{3}{5}} = 0 \\ u_1' + \frac{3}{5}u_2'x^{-\frac{2}{5}} = \frac{1}{5}x^{-\frac{3}{2}}. \end{cases}$$

$$W(x, y_1, y_2) = \begin{vmatrix} x & x^{\frac{3}{5}} \\ 1 & \frac{3}{5}x^{-\frac{2}{5}} \end{vmatrix} = -\frac{2}{5}x^{\frac{3}{5}},$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & x^{\frac{3}{5}} \\ \frac{1}{5}x^{-\frac{3}{2}} & \frac{3}{5}x^{-\frac{2}{5}} \end{vmatrix} = -\frac{1}{5}x^{-\frac{9}{10}},$$

$$W_2(x, y_1, y_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{1}{5}x^{-\frac{3}{2}} \end{vmatrix} = \frac{1}{5}x^{-\frac{1}{2}},$$

then

$$u_1' = \frac{W_1}{W} = \frac{1}{2}x^{-\frac{3}{2}},$$

and

$$u_1 = -x^{-\frac{1}{2}}.$$

But

$$u_2' = \frac{W_2}{W} = -\frac{1}{2}x^{\frac{11}{10}},$$

then

$$u_2 = 5x^{-\frac{1}{10}}.$$

Thus

$$y_p = y_1u_1 + y_2u_2 = x(-x^{-\frac{1}{2}}) + x^{\frac{3}{5}}(5x^{-\frac{1}{10}}) = 4x^{\frac{1}{2}}.$$

We can find the particular solution of (19) by substituting $y_p = A\sqrt{x}$ and it is easy to see that $A = 4$. Then the general solution of (19) is

$$y = y_c + y_p = c_1x + c_2x^{\frac{3}{5}} + 4x^{\frac{1}{2}}.$$

Example (9) Find the general solution of the differential equation

$$y''' + y' = \tan x \quad ; \quad 0 < x < \frac{\pi}{2}. \tag{21}$$

Solution

1) The the general solution of

$$y''' + y' = 0,$$

is

$$y_c = c_1 + c_2\cos x + c_3\sin x.$$

2) Let $y_1 = 1$, $y_2 = \cos x$ and $y_3 = \sin x$. The particular solution 0f (21) is the form

$$y_p = u_1y_1 + u_2y_2 + u_3y_3,$$

where u_1, u_2 and u_3 are three functions satisfying the system

$$\begin{cases} u_1'(1) + u_2'\cos x + u_3'\sin x = 0 \\ -u_2'\sin x + u_3'\cos x = 0 \\ -u_2'\cos x - u_3'\sin x = \tan x. \end{cases}$$

We have

$$W(x, y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1,$$

$$W_1(x, y_1, y_2, y_3) = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x\ ,$$

$$W_2(x, y_1, y_2, y_3) = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = -\sin x \ ,$$

$$W_3(x, y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = \frac{-\sin^2(x)}{\cos x}.$$

Then we have

$$u_1' = \frac{W_1}{W} = \tan x,$$

and

$$u_1 = \int \tan x dx = -\ln(\cos x) \ .$$

But

$$u_2' = \frac{W_2}{W} = -\sin x \ ,$$

then

$$u_2 = -\int \sin x dx = \cos x.$$

Also

$$u_3' = \frac{W_3}{W} = \frac{-\sin^2(x)}{\cos x},$$

hence

$$u_3 = -\int \frac{\sin^2(x)}{\cos x} dx = -\int \frac{1 - \cos^2(x)}{\cos x} dx = -\ln(\sec x + \tan x) + \sin x.$$

Thus

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 + u_3 y_3, \\ &= -\ln(\cos x) + \cos^2(x) - \sin x \ln(\sec x + \tan x) + \sin^2(x), \\ &= 1 - \ln(\cos x) - \sin x \ln(\sec x + \tan x). \end{aligned}$$

So the general solution of (21) is

$$\begin{aligned} y &= y_c + y_p = (c_1 + 1) + c_2 \cos x + c_3 \sin x - \ln(\cos x) - \sin x \ln(\sec x + \tan x) \\ &= y = c_4 + c_2 \cos x + c_3 \sin x - \ln(\cos x) - \sin x \ln(\sec x + \tan x). \end{aligned}$$

Example (10) Find the general solution of the differential equation

$$x^3y''' + x^2y'' - 2xy' + 2y = 2x^4 \quad ; \quad x > 0. \tag{22}$$

Solution

1) First we have to find the general solution of

$$x^3y''' + x^2y'' - 2xy' + 2y = 0. \tag{23}$$

By substituting $y = x^m$, where m is a root of the characteristic equation

$$m(m-1)(m-2) + m(m-1) - 2m + 2 = (m-1)(m-2)(m+1) = 0.$$

Then we have three roots real and distinct $m = 1$, $m = 2$, and $m = -1$, hence the general solution of (23) is

$$y_c = c_1x + c_2x^2 + c_3x^{-1} \ ; \quad x > 0.$$

2) Let $y_1 = x$, $y_2 = x^2$ and $y_3 = x^{-1}$. The particular solution of (22) is the form

$$y_p = u_1y_1 + u_2y_2 + u_3y_3 \ ,$$

where u_1 , u_2 and u_3 are three functions satisfying the system

$$\begin{cases} u_1'x + u_2'x^2 + u_3'x^{-1} = 0 \\ u_1' + 2u_2'x - u_3'x^{-2} = 0 \\ 2u_2' + 2u_3'x^{-3} = 2x. \end{cases}$$

We have

$$W(x, y_1, y_2, y_3) = \begin{vmatrix} x & x^2 & x^{-1} \\ 1 & 2x & -x^{-2} \\ 0 & 2 & 2x^{-3} \end{vmatrix} = 6x^{-1},$$

$$W_1(x, y_1, y_2, y_3\ ) = \begin{vmatrix} 0 & x^2 & x^{-1} \\ 0 & 2x & -1x^{-2} \\ 2x & 2 & 2x^{-3} \end{vmatrix} = -6x,$$

$$W_2(x, y_1, y_2, y_3) = \begin{vmatrix} x & 0 & x^{-1} \\ 1 & 0 & -1x^{-2} \\ 0 & 2x & 2x^{-3} \end{vmatrix} = 4,$$

$$W_3(x, y_1, y_2, y_3) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 2x \end{vmatrix} = 2x^3.$$

Then we have

$$u_1' = \frac{W_1}{W} = -x^2,$$

and

$$u_1 = -\frac{1}{3}x^3.$$

But

$$u_2' = \frac{W_2}{W} = \frac{2}{3}x,$$

hence

$$u_2 = \frac{1}{3}x^2,$$

Also

$$u_3' = \frac{W_3}{W} = \frac{1}{3}x^4,$$

then

$$u_3 = \frac{1}{15}x^5.$$

Then

$$y_p = u_1y_1 + u_2y_2 + u_3y_3 = -\frac{1}{3}x^4 + \frac{1}{3}x^4 + \frac{1}{15}x^4 = \frac{1}{15}x^4,$$

and the general solution of (22) is

$$y = y_c + y_p = c_1x + c_2x^2 + c_3x^{-1} + \frac{1}{15}x^4 \quad ; \quad x > 0.$$

Example (11) Find the solution of the initial value problem (IVP)

$$\begin{cases} 2x^2y'' + xy' - 3y = x^{-3} & ; \quad x > 0 \\ y(1) = 1 \quad , \quad y'(1) = -1. \end{cases} \tag{24}$$

Solution

1) We have to find the general solution of

$$2x^2y'' + xy' - 3y = 0.$$

By substituting $y = x^m$, we have

$$m(m-1) + m - 3 = (2m-3)(m+1) = 0,$$

hence the general solution of the homogeneous differential equation is

$$y_c = c_1 \, x^{-1} + c_2 \, x^{\frac{3}{2}}.$$

2) Let $y_1 = x^{-1}$, $y = x^{\frac{3}{2}}$, then

$$y_p = u_1 y_1 + u_2 y_2,$$

where u_1 and u_2 are two functions satisfying the system

$$\begin{cases} u_1' x^{-1} + u_2' x^{\frac{3}{2}} = 0 \\ u_1'(-x^{-2}) + u_2'(\frac{3}{2} x^{\frac{1}{2}}) = \frac{1}{2} x^{-5}, \end{cases}$$

$$W(x, y_1, y_2) = \begin{vmatrix} x^{-1} & x^{\frac{3}{2}} \\ -x^{-2} & \frac{3}{2} x^{\frac{1}{2}} \end{vmatrix} = \frac{5}{2} x^{-\frac{1}{2}},$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & x^{\frac{3}{2}} \\ \frac{1}{2} x^{-5} & \frac{3}{2} x^{\frac{1}{2}} \end{vmatrix} = -\frac{1}{2} x^{-\frac{7}{2}},$$

$$W_2(x, y_1, y_2) = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{1}{2} x^{-5} \end{vmatrix} = \frac{1}{2} x^{-6}.$$

Then we have

$$u_1' = \frac{W_1}{W} = -\frac{1}{5} x^{-3},$$

and

$$u_1 = \frac{1}{10} x^{-2}.$$

But

$$u_2' = \frac{W_2}{W} = \frac{1}{5} x^{-\frac{11}{2}},$$

hence

$$u_2 = -\frac{2}{45} x^{-\frac{9}{2}}.$$

So

$$y_p = u_1 y_1 + u_2 y_2 = \frac{1}{10} x^{-3} - \frac{2}{45} x^{-3} = \frac{1}{18} x^{-3}.$$

Then the general solution of (24) is

$$y = y_c + y_p = c_1 \, x^{-1} + c_2 \, x^{\frac{3}{2}} + \frac{1}{18} x^{-3}.$$

We can obtain y_p by substituting $y_p = Ax^{-3}$, which implies $A = \frac{1}{18}$.

3)

$$y'(x) = -c_1x^{-2} + \frac{3}{2}c_2x^{\frac{1}{2}} - \frac{1}{6}x^{-4}.$$

From the conditions $y(1) = 1$ and $y'(1) = -1$, we deduce

$$c_1 + c_2 = \frac{17}{18},$$

and

$$-c_1 + \frac{3}{2}c_2 = -\frac{5}{6},$$

which implies $c_1 = \frac{9}{10}$ and $c_2 = \frac{2}{45}$. Thus the solution of the IVP is

$$y = \frac{9}{10}x^{-1} + \frac{2}{45}x^{\frac{3}{2}} + \frac{1}{18}x^{-3}.$$

Exercises (4.6)

In exercises 1 through 33 use the variation of parameters method to compute the general solution or initial value problems of the nonhomogeneous differential equations.

1) $y'' + y = \tan x \quad ; \quad 0 < x < \frac{\pi}{2}$.
2) $y'' + y = \sec x \quad ; \quad 0 < x < \frac{\pi}{2}$.
3) $y'' - 2y' + y = \frac{1}{x}e^x \quad ; \quad x > 0$.
4) $y'' + 10y' + 25y = e^{-5x}\frac{\ln x}{x^2} \quad ; \quad x > 0$.
5) $y'' + 6y' + 9y = \frac{1}{x^3}e^{-3x} \quad ; \quad x > 0$.
6) $y'' - 12y' + 36y = e^{6x}\ln x \quad ; \quad x > 0$.
7) $y'' + y = \csc x . \cot x \quad ; \quad 0 < x < \frac{\pi}{2}$.
8) $y'' + 4y' + 5y = e^{-2x}\sec x \quad ; \quad 0 < x < \frac{\pi}{2}$.
9) $y'' + y = \sec^3(x) \quad ; \quad 0 < x < \frac{\pi}{2}$.
10) $y'' - 4y' + 4y = e^{2x}x^{-4} \quad ; \quad x > 0$.
11) $y'' + 2y' + y = x^{-2}e^{-x}\ln x \quad ; \quad x > 0$.
12) $y'' - 2y' + y = \frac{e^x}{(e^x+1)^2}$.
13) $y'' + 2y' + 2y = e^{-x}\csc x \quad ; \quad 0 < x < \frac{\pi}{2}$.
14) $y'' + y = \tan^2(x) \quad ; \quad 0 < x < \frac{\pi}{2}$.
15) $y'' + y = \sec^2(x). \csc x \quad ; \quad 0 < x < \frac{\pi}{2}$.
16) $y'' - 3y' + 2y = \cos(e^{-x})$.

17) $y'' - y = \frac{2}{\sqrt{1-e^{-2x}}} \quad ; \quad x > 0$.

18) $y'' - y = e^{-2x}\sin(e^{-x})$.

19) $5x^2y'' - 3xy' + 3y = x^{\frac{1}{2}}$; $x > 0$.

20) $x^2y'' + 4xy' - 4y = x^{\frac{1}{4}}\ln x$; $x > 0$.

21) $2x^2y'' + xy' - 3y = x^{-3}$; $x < 0$.

22) $2x^2y'' + 7xy' - 3y = x^{-2}\ln x$; $x > 0$.

24) $y''' + y' = \tan x$; $0 < x < \frac{\pi}{2}$.

25) $y''' + 4y' = \sec(2x)$; $0 < x < \frac{\pi}{4}$.

26) $2y''' - 6y'' = x^2$..

27) $\begin{cases} y'' + y = \csc x \\ y(\frac{\pi}{4}) = 0 \ , \ y'(\frac{\pi}{4}) = 1. \end{cases}$

28) $\begin{cases} y'' + y = \tan x \\ y(\frac{\pi}{3}) = 1 \ , \ y'(\frac{\pi}{3}) = 0. \end{cases}$

29) $\begin{cases} y'' - 2y' + y = \frac{e^x}{x} \\ y(1) = e \ , \ y'(1) = 0. \end{cases}$

30) $\begin{cases} y'' + 6y' + 9y = x^{-3}e^{-3x} \\ y(1) = 4e^{-3} \ , \ y'(1) = -2e^{-3}. \end{cases}$

31) $\begin{cases} y'' + y = \sec^3(x) \\ y(0) = 1 \ , \ y'(0) = 1. \end{cases}$

32) $\begin{cases} y'' - 4y' + 4y = x^{-4}e^{2x} \\ y(1) = 0 \ , \ y'(1) = e^2. \end{cases}$

33) $\begin{cases} y'' - 2y' + y = \frac{e^{2x}}{(e^x+1)^2} \\ y(0) = 3 \ , \ y'(0) = \frac{5}{2}. \end{cases}$

34) Given that $y_1 = x^{-\frac{1}{2}}\cos x$ and $y_2 = x^{-\frac{1}{2}}\sin x$; $x > 0$, form a fundamental set of solutions of

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0 \ \ on \ \ (0, \infty).$$

Find the general solution of

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{\frac{3}{2}} \ \ ; \ x > 0.$$

35) Find the general solution of the differential equation by two methods

$$y''' - 5y'' + 6y' = 2\sin x + 8,$$

REVIEW EXERCISES

In exercises 1 through 25, they are multiple choice, each exercise has four solutions and there is only one correct answer, find it.

1) The initial value problem

$$\begin{cases} y'' + 9y = 0 \\ y(0) = 0\ ,\ y'(0) = 0. \end{cases}$$

a) has a unique solution.
b) may have no solution.
c) may have many solution.
d) b) and c) are possible.

2) A fundamental set of solutions of a homogeneous linear differential equations of order n comprises

a) any set of solutions.
b) any set of n solutions.
c) any set of n solutions linearly independent.
d) any set of n solutions linearly independent.

3) To obtain the general solution of a homogeneous linear differential equation of order n, we must construct a linear combination of

a) any set of solutions.
b) any set of n solutions.
c) any set of n solutions linearly independent.
d) any set of n homogeneous solutions linearly independent.

4) If $y_1(x)$, $y_2(x)$,......, $y_n(x)$ form a fundamental set of solutions of a homogeneous linear differential equations of order n on an interval I , then the *Wronskian* $W(y_1(x)\ ,\ y_2(x)\ ,......,\ y_n(x))$

a) is zero at every $x \in I$.
b) may be zero at infinity many $x \in I$.
c) is not zero at any $x \in I$.
d) is not zero at any $x \in I$ except at $x = 0$.

5) If a function

$$y = c_1y_1(x) + c_2y_2(x) + c_3y_3(x),$$

is a solution of a homogeneous linear differential equations of order 2 on an interval I , then $y_1(x)$, $y_2(x)$ and $y_3(x)$

a) form fundamental set of solutions of the differential equation on I .
b) are linearly independent on I.

c) are linearly dependent on I.

d) need not all be solutions of the differential equation on I.

6) If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the same second order differential equation, then $\frac{y_1}{y_2}$

a) is a function of x.

b) is a constant,

c) could be a) or b).

d) none of these.

7) If $y_1 = e^{2x}$ is a solution of the given differential equation, use the reduction of order to find a second solution y_2 of

$$y'' - 5y' + 6y = 0$$

a) $y_2 = e^{-2x}$, b) $y_2 = xe^{2x}$, c) $y_2 = e^{-3x}$, d) $y_2 = e^{3x}$.

8) If $y_1 = x^2$ is a solution of the given differential equation

$$x^2y'' - 3xy' + 4y = 0 \ ; \ x > 0,$$

, then the second solution is

a) $y_2 = x^{-2}\ln x$, b) $y_2 = x^{-2}$, c) $y_2 = x^2 \ln x$, d) $y_2 = x^4$.

9) If an initial value problem comprises a 4th-order homogeneous linear differential equations , we would expect it to have

a) one initial condition.

b) two initial conditions.

c) three initial conditions.

d) eight initial conditions.

10) If $y = xe^{-3x}$ is a solution of homogeneous linear differential equation , then another solution must be

a) $y = x$, b) $y = e^{-3x}$, c) $y = xe^{3x}$, d) $y = x^2e^{-3x}$.

11) If $y = x\cos(2x)$ is a solution of homogeneous linear differential equation , then another solution must be

a) $y = x\sin(2x)$, b) $y = x\sin x$, c) $y = \sin(2x)$, d) all of the above.

12) The roots of the characteristic (auxiliary) equation of a homogeneous linear differential equation :

$$x^2y'' + xy' + y = 0,$$

is a) $-1, +1$. b) $+i$, $-i$. c) $1+i$, $1-i$. d) None of these.

13) If the characteristic equation of a homogeneous linear-differential equation has a factor $(m-4)^3$, then the solutions of the differential equation must include

a) $3+7i$, b) $-3+7i$, c) $3-7i$, d) $-3-7i$.

a) $y=e^{4x}$, b) $y=e^{4x}$ and $y=xe^{4x}$, c) $y=xe^{4x}$ and $y=x^2e^{4x}$, d) $y=e^{4x}$, $y=xe^{4x}$ and $y=x^2e^{4x}$.

14) It is possible for the characteristic equation of 3rd-order homogeneous linear differential equation to have

a) two real roots and one complex root.

b) one real and two complex roots.

c) three complex roots.

d) a) or c).

15) undetermined coefficients cannot be used if the input function contains what term?

a) e^{3x} , b) $\ln(4x)$, c) x^2 , d) $x\cos(4x)$.

16) undetermined coefficients cannot be used if the input function contains what term?

a) $\sqrt{x}$, b) x^2e^{-3x} , c) $x^2\cos(2x)$, d) $x^3\sin(4x)$.

17) Without solving the differential equation, apply undetermined coefficients the simplest form of a particular solution of the differential

$$y''-16y=e^{4x}.$$

a) $y_p=ae^{4x}$, b) $y_p=axe^{4x}$, c) $y_p=a\sin(4x)$, d) $y_p=a\cos(4x)$.

18) Without solving the differential equation, apply undetermined coefficients the simplest form of a particular solution of the differential equation

$$y''+9y=4\cos(3x).$$

a) $y_p=a\cos(3x)+b\sin(3x)$.

b) $y_p=ax\cos(3x)+b\sin(3x)$.

c) $y_p=ax\cos(3x)+bx\sin(3x)$.

d) $y_p=a\cos(3x)+bx\sin(3x)$.

19) We use the variation of parameters to obtain

a) a particular solution of a nonhomogeneous differential equation.

b) the complementary solution of linear differential equation.

c) both a) and b).

d) neither a) nor b).

20) When using variation of parameters to solve a 2nd-order linear differential equation, we must evaluate

a) 1 integral , b) 2 integrals , c) 3 integrals , d) 4 integrals.

21) Solving a *Euler's* equation equivalent to using the substitution , where $x > 0$.

a) $y = m^x$, b) $y = \ln x$, c) $y = x^m$, d) $y = e^{mx}$.

22) We may transform a *Euler* differential equation to a *Euler* differential equation with constant coefficients by using the variable substituting , where $x > 0$.

a) $x = e^t$, b) $x = t^m$, c) $t = e^x$, d) $t = x^m$.

23) If the characteristic equation of a homogeneous *Euler* differential equation is $(m-3)^2 = 0$, then linearly independent solutions of the differential equation are (where $x > 0$)

a) $y_1 = x^3$ and $y_2 = x^4$.

b) $y_1 = x^3$ and $y_2 = x^3 \ln x$.

c) $y_1 = e^{3x}$ and $y_2 = xe^{3x}$.

d) $y_1 = e^{3x}$ and $y_2 = e^{3x} \ln x$.

24) If the characteristic equation of a homogeneous *Euler* differential equation is $m^2(m-4) = 0$, then linearly independent solutions of the differential equation are

a) $y_1 = 1$, $y_2 = x$, and $y_3 = e^{4x}$.

b) $y_1 = 1$, $y_2 = \ln x$, and $y_3 = e^{4x}$.

c) $y_1 = 1$, $y_2 = x$ and $y_3 = x^4$.

d) $y_1 = 1$, $y_2 = \ln x$ and $y_3 = x^4$.

25) To obtain a solution of the *Euler* differential equation that is valid on the interval $(-\infty, 0)$ we must use the substitution

a) $-x = \ln t$, b) $x = t^m$, c) $x = t^{-m}$, d) $t = \ln(-x)$.

Complete the statement of the exercises 26 through 33.

26) The only solution of $y'' + x^2 y = 0$, $y(0) = 0$, $y'(0) = 0$ is ________________ .

27) If two differential functions f_1 and f_2 are linearly independent on an interval I , then $W(f_1(x), f_2(x)) \neq 0$ for at least one point in the interval I __________.

28) The functions $f_1(x) = x^2$, $f_2(x) = 1 - x^2$ and $f_3(x) = 2 + x^2$ are linearly on the interval ________________ .

29) The functions $f_1(x) = x^2$ and , $f_2(x) = x\,|x|$ linearly independent on the interval ____________ , whereas they are linearly dependent on the interval ________________ .

30) Two solutions y_1 and y_2 of $y'' + y' + y = 0$ are linearly dependent if $W(y_1, y_2) = 0$ for every real value x __________ .

31) A constant multiple of a solution of a differential equation is also a solution __________ .

32) A fundamental set of solutions of $(x-2)y'' + y = 0$ exist on any interval not containing the point ________ .

33) For the method of undetermined coefficients , the assumed form of the particular solution y_p for $y'' - y = 1 + e^x$ is ________ .

In exercises 34 through 35, write the *Wronskian* of two linearly dependent solutions without finding the solutions themselves.

34) $2y'' + \frac{7}{x}y' - y = 0.$

35) $y'' - 8y' + 8e^x y = 0.$

36) Show that $W(5, \sin^2 x, \cos(2x)) = 0$ for all x .Can you establish this result directly without evaluating the *Wronskian* ?

37) Verify that there exist a unique solution to the initial value problem

$$\begin{cases} (2-x)y'' + (\cos x)y' + (\tan x)y = 0 \quad ; \quad -1 < x < 1 \\ y(0) = 0 \quad , \quad y'(0) = 1. \end{cases}$$

38) Find the second solution for the differential equation given that $y_1(x)$ is known solution.

$$xy'' - 2(x+1)y' + (x+2)y = 0 \quad ; \quad y_1 = e^x.$$

39) A certain homogeneous linear differential equation with constant coefficients has the characteristic equation

$$(m-7)^3(m^2-4)(m^2+2m+2) = 0.$$

Write the general solution of this differential equation and state its order.

In problems 40 through 65 find the general solution of each differential equation.

40) $y'' - 2y' - 2y = 0.$

41) $2y'' + 2y' + 3y = 0.$

42) $y''' + 10y'' + 25y' = 0.$

43) $3y''' + 10y'' + 15y' + 4y = 0.$

44) $2y''' + 9y'' + 12y' + 5y = 0.$

45) $y^{(4)} + 3y''' + 2y'' + 6y' - 4y = 0.$

46) $y'' + 17y' + 16y = e^x + 4e^{-x}.$

47) $(x^2-2x)y''+(2-x^2)y'+2(x-1)y=0$; $0<x<1$, where $y_1=x^2$ is a particular solution.

48) $3x^2y''-2xy'-12y=0$; $x>0$.

49) $y''+y=\csc x+1$; $0<x<1$.

50) $y''-y=x^3e^x$.

51) $x^2y''+7xy'+9y=0$; $x<0$.

52) $y'''+27y=x$.

53) $9y''+48y'+64y=\sin(2x)$.

54) $y''+y=\cot x$; $0<x<\pi$.

55) $(x\cos x-\sin x)y''+(x\sin x)y'-(\sin x)y=0$; $\frac{\pi}{4}<x<\frac{\pi}{2}$, where $y_1=\sin x$ is a particular solution.

56) $y''+y=x\sin x$.

57) $y^{(4)}-2y'''+2y''-2y'+y=0$.

58) $x^2y''+9xy'+17y=0$; $x>0$.

59) $3x^2y''-2xy'-12y=x^{-2}$; $x>0$.

60) $y''-3y'+5y=4x^3-2x$.

61) $y'''-5y''+6y'=2\sin x+8$.

62) $y''-2y'+y=x^2e^{2x}$.

63) $y'''-y''=6$.

64) $y''-2y'+2y=e^x\tan x$; $0<x<\frac{\pi}{2}$.

65) $y''-y=\frac{2e^x}{e^x+e^{-x}}$.

In problems 66 through 77, solve the given differential equation subject to the indicate conditions .

66) $\begin{cases} y''-2y'+2y=0 \\ y(\frac{\pi}{2})=0\ ,\ \ y'(\frac{\pi}{2})=-1. \end{cases}$

67) $\begin{cases} y''-y=x+\sin x \\ y(0)=2\ ,\ y'(0)=3. \end{cases}$

68) $\begin{cases} y''+y=\sec^3(x) \\ y(0)=1\ ,\ \ y'(0)=\frac{1}{2}. \end{cases}$

69) $\begin{cases} y'''-4y'=0 \\ y(0)=0,\ \ y'(0)=0,\ y''(0)=2. \end{cases}$

70) $\begin{cases} y^{(4)}+2y''+y=3x+4 \\ y(0)=y'(0)=0\ ,\ y''(0)=y'''(0)=1. \end{cases}$

71) $\begin{cases} y'''-3y''+2y'=x+e^x \\ y(0)=1,\ y'(0)=-\frac{1}{4}\ ,y''(0)=-\frac{3}{2}. \end{cases}$

72) $\begin{cases} x^2y''+4xy'=0 \\ y(1)=0\ ,\ y'(1)=6 \end{cases}$

73) $\begin{cases} x^2y'' - 5xy' + 8y = 0 \\ y(2) = 32 \ , \ y'(2) = 0 \end{cases}$

74) $\begin{cases} xy'' + y' = x \\ y(1) = 1 \ , \ y'(1) = -\frac{1}{2} \end{cases}$

75) $\begin{cases} x^2y'' - 5xy' + 8y = 8x^6 \\ y(\frac{1}{2}) = 0 \ , \ y'(\frac{1}{2}) = 0. \end{cases}$

76) $\begin{cases} 4x^2 + y = 0 \\ y(-1) = 2 \ , \ y'(-1) = 4. \end{cases}$

77) $\begin{cases} x^2y'' - 4xy' + 6y = 0 \\ y(-2) = 8 \ , \ y'(-2) = 0. \end{cases}$

78) Show that the general solution of the differential equation $y^{(4)} - y = 0$ can be written as

$$y = c_1 \cos x + c_2 \sin x + c_3 \cosh x + c_4 \sinh x \ .$$

79) Find the particular solution of

$$y''' - 4y' = x + 3\cos x + e^{-2x}.$$

80) Find the general solution of the differential equation

$$(x+2)^2y'' + (x+2)y' + y = 0.$$