Integral Calculus

Department of Mathematics

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Main Contents

- Antiderivatives.
- Indefinite integrals.
- Main properties of indefinite integrals.
- Substitution method.

Antiderivatives

Definition

A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x)$$
 for every $x \in I$.



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Antiderivatives

Definition

A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x)$$
 for every $x \in I$.

Example

• Let
$$F(x) = x^2 + 3x + 1$$
 and $f(x) = 2x + 3$.
Since $F'(x) = f(x)$, then the function $F(x)$ is an antiderivative of $f(x)$.

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Theorem

If functions F and G are antiderivatives of f on an interval I, there exists a constant c such that G(x) = F(x) + c for every $x \in I$.

Example

Let f(x) = 2x. The functions $F(x) = x^2 + 2$, $G(x) = x^2 - \frac{1}{2}$, $H(x) = x^2 - \sqrt[3]{2}$, are antiderivatives of the function f. Therefore, $F(x) = x^2 + c$ is the general form of the antiderivatives (the family) of the function f(x) = 2x.

Example

Find the general form of the antiderivatives of $f(x) = 6x^5$.

Solution:

The function $F(x) = x^6 + c$ is the general antiderivative of f because $F'(x) = 6x^5$.

Indefinite Integrals

Definition

Let f be a continuous function on an interval I. The indefinite integral of f is the general antiderivative of f on I:

$$\int f(x) \, dx = F(x) + c.$$

The function f is called the integrand, the symbol $\int f$ is the integral sign, x is called the variable of the integration and c is the constant of the integration.

Now, by using the previous definition, the general antiderivatives in the previous example are

1
$$\int (2x+3) \, dx = x^2 + 3x + c.$$

 2 $\int (\cos x + 1) \, dx = \sin x + x + c.$

Basic Integration Rules

Rule 1: Power of x.

$$\frac{d}{dx}(\frac{x^{n+1}}{n+1}) = x^n$$
, so $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$

In words, to integrate the function x^n , we add 1 to the power (i.e., n + 1) and divide the function by n + 1. If n = 1, we have a special case

$$\int 1 \, dx = x + c.$$

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Rule 2: Trigonometric functions.

$$\frac{d}{dx}\sin x = \cos x, \text{ so } \int \cos x \, dx = \sin x + c$$
$$\frac{d}{dx}\cos x = -\sin x, \text{ so } \int -\sin x \, dx = \cos x + c$$
Therefore, $\int \sin x \, dx = -\cos x + c$.

The other trigonometric functions with the previous rules are listed in the following table:

Derivative	Indefinite Integral
$\frac{d}{dx}(x) = 1$	$\int 1 dx = x + c$
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, \ n \neq -1$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x dx = \sin x + c$
$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x dx = \tan x + c$
$\frac{d}{dx}(-\cot x) = \csc^2 x$	$\int \csc^2 x dx = -\cot x + c$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \ \tan x \ dx = \sec x + c$
$\frac{d}{dx}(-\csc x) = \csc x \cot x$	$\int \csc x \cot x dx = -\csc x + c$

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Evaluate the integral.

$$\int x^{-3} dx$$

$$\int \frac{1}{\cos^2 x} dx$$



Image: Image:

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Solution:

$$\int x^{-3} dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c.$$

Image: A matrix

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Solution:

$$\int x^{-3} dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c.$$

$$\int \frac{1}{\cos^2 x} dx = \int \sec^2 x \, dx = \tan x + c.$$

$$(\sec x = \frac{1}{\cos x} \Rightarrow \sec^2 x = \frac{1}{\cos^2 x})$$

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Properties of the Indefinite Integral

Theorem

Assume f and g have antiderivatives on an interval I, then

1
$$\frac{d}{dx} \int f(x) \, dx = f(x).$$

 2 $\int \frac{d}{dx} (F(x)) \, dx = F(x) + c.$

 3 $\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx.$

 3 $\int kf(x) \, dx = k \int f(x) \, dx, \text{ where } k \text{ is a constant}$

Example

Evaluate the integral.

1
$$\int (4x+3) dx$$

2 $\int (2\sin x + 3\cos x) dx$
3 $\int (\sqrt{x} + \sec^2 x) dx$

$$\int \frac{d}{dx} (\sin x) dx$$

$$\int \frac{d}{dx} \int \sqrt{x+1} dx$$

Solution:

1
$$\int (4x+3) \, dx = \frac{4x^2}{2} + 3x + c = 2x^2 + 3x + c.$$

 2 sin x + 3 cos x) $dx = -2 \cos x + 3 \sin x + c.$

 3 $\int (\sqrt{x} + \sec^2 x) \, dx = \frac{x^{\frac{3}{2}}}{3/2} + \tan x + c = \frac{2x^{\frac{3}{2}}}{3} + \tan x + c.$

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Example

If
$$\int f(x) dx = x^2 + c_1$$
 and $\int g(x) dx = \tan x + c_2$, find $\int (3f(x) - 2g(x)) dx$.

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Example

If
$$\int f(x) dx = x^2 + c_1$$
 and $\int g(x) dx = \tan x + c_2$, find $\int (3f(x) - 2g(x)) dx$.

Solution:

From the third and fourth properties, $\int (3f(x) - 2g(x)) dx = 3 \int f(x) dx - 2 \int g(x) dx = 3x^2 - 2 \tan x + c, \text{ where } c = 3c_1 - 2c_2.$

Solve the differential equation $f'(x) = x^3$ subject to the initial condition f(0) = 1.



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Solution:

$$\int f'(x) \, dx = \int x^3 \, dx$$
$$\Rightarrow f(x) = \frac{1}{4}x^4 + c.$$

If x = 0, then $f(0) = \frac{1}{4}(0)^4 + c = 1$ and this implies c = 1. Hence, the solution of the differential equation is $f(x) = \frac{1}{4}x^4 + 1$.

Solve the differential equation $f'(x) = x^3$ subject to the initial condition f(0) = 1.

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When evaluating integrals, we can always check our answers by deriving the results. In the previous examples, we use x as a variable of the integration. However, for this role, we can use any variable such as y, z, t, etc. That is, instead of f(x) dx, we can integrate f(y) dy or f(t) dt.

Integration By Substitution

Theorem

Let g be a differentiable function on an interval I where the derivative is continuous. Let f be continuous on the interval J contains the range of the function g. If F is an antiderivative of the function f on J, then

$$\int f(g(x))g'(x) \, dx = F(g(x)) + c, \ x \in I.$$

Example

Evaluate the integral $\int 2x (x^2 + 1)^3 dx$.

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$$\int f(g(x))g'(x) \, dx = F(g(x)) + c, \ x \in I.$$

Example

Evaluate the integral
$$\int 2x (x^2 + 1)^3 dx$$
.

Solution:

We can use the previous theorem as follows:

let $f(x) = x^3$ and $g(x) = x^2 + 1$, then $f(g(x)) = (x^2 + 1)^3$. Since g'(x) = 2x, then from the theorem, we have

$$\int 2x(x^2+1)^3 \, dx = \frac{(x^2+1)^4}{4} + c.$$

We can end with the same solution by using the five steps of the substitution method given below.

Steps of the integration by substitution:

- *Step 1:* Choose a new variable *u*.
- Step 2: Determine the value of du.
- Step 3: Make the substitution i.e., eliminate all occurrences of x in the integral by making the entire integral in terms of u.
- *Step 4:* Evaluate the new integral.
- *Step 5:* Return the evaluation to the initial variable *x*.

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- *Step 4:* Evaluate the new integral.
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In the previous example, let $u = x^2 + 1$, then du = 2x dx. By substituting that into the original integral, we have

$$\int u^3 \, du = \frac{u^4}{4} + c.$$

Now, by returning the evaluation to the initial variable x, we have $\int 2x(x^2+1)^3 dx = \frac{(x^2+1)^4}{4} + c.$

Evaluate the integral
$$\int \frac{se}{r}$$

$$\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx.$$



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valuate the integral
$$\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx.$$

Solution:

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We use the theorem for the integral $2\int \frac{\sec^2 \sqrt{x}}{2\sqrt{x}} dx$. Let $f(x) = \sec^2 x$ and $g(x) = \sqrt{x}$, then $f(g(x)) = \sec^2 \sqrt{x}$. Since $g'(x) = 1/(2\sqrt{x})$, then we have

$$\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx = 2 \tan \sqrt{x} + c.$$

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$$\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx = 2 \tan \sqrt{x} + c.$$

By using the steps of the substitution method, let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}} dx$. By substitution, we obtain

$$2\int \sec^2 u \, du = 2\tan u + c = 2\tan \sqrt{x} + c.$$

Evaluate the integral
$$\int \frac{x^2-1}{(x^3-3x+1)^6} dx$$
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Evaluate the integral
$$\int rac{x^2-1}{(x^3-3x+1)^6} \ dx.$$

Solution:

Let $u = x^3 - 3x + 1$, then $du = 3(x^2 - 1) dx$. By substitution, we have

$$\frac{1}{3}\int u^{-6} du = \frac{1}{3} \frac{1}{-5u^5} + c = \frac{-1}{15(x^3 - 3x + 1)^5} + c$$

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The upcoming corollary simplifies the process of the substitution method for some functions.

Corollary

If
$$\int f(x) dx = F(x) + c$$
, then for any $a \neq 0$,
 $\int f(ax \pm b) dx = \frac{1}{a}F(ax \pm b) + c.$

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Evaluate the integral.

$$\int \sqrt{2x-5} \, dx$$

$$\int \cos (3x+4) \, dx$$



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