# Integral Calculus 

Department of Mathematics

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## Chapter 1: The Indefinite Integrals

## Main Contents

(1) Antiderivatives.
(2) Indefinite integrals.
(3) Main properties of indefinite integrals.
(4) Substitution method.

## Antiderivatives

## Definition

A function $F$ is called an antiderivative of $f$ on an interval l if

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F^{\prime}(x)=f(x) \text { for every } x \in I
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## Antiderivatives

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## Example

(1) Let $F(x)=x^{2}+3 x+1$ and $f(x)=2 x+3$.

Since $F^{\prime}(x)=f(x)$, then the function $F(x)$ is an antiderivative of $f(x)$.
(2) Let $G(x)=\sin x+x$ and $g(x)=\cos x+1$.

Since $G^{\prime}(x)=\cos x+1$, then the function $G(x)$ is an antiderivative of $g(x)$.

## Theorem

If functions $F$ and $G$ are antiderivatives of $f$ on an interval I, there exists a constant $c$ such that $G(x)=F(x)+c$ for every $x \in I$.

## Example

Let $f(x)=2 x$. The functions
$F(x)=x^{2}+2$,
$G(x)=x^{2}-\frac{1}{2}$,
$H(x)=x^{2}-\sqrt[3]{2}$,
are antiderivatives of the function $f$. Therefore, $F(x)=x^{2}+c$ is the general form of the antiderivatives (the family) of the function $f(x)=2 x$.

## Example

Find the general form of the antiderivatives of $f(x)=6 x^{5}$.
Solution:
The function $F(x)=x^{6}+c$ is the general antiderivative of $f$ because $F^{\prime}(x)=6 x^{5}$.

## Indefinite Integrals

## Definition

Let $f$ be a continuous function on an interval $l$. The indefinite integral of $f$ is the general antiderivative of $f$ on $I$ :

$$
\int f(x) d x=F(x)+c
$$

The function $f$ is called the integrand, the symbol $\int$ is the integral sign, $x$ is called the variable of the integration and $c$ is the constant of the integration.

Now, by using the previous definition, the general antiderivatives in the previous example are
(1) $\int(2 x+3) d x=x^{2}+3 x+c$.
(2) $\int(\cos x+1) d x=\sin x+x+c$.

## Basic Integration Rules

$\square$ Rule 1: Power of $x$.

$$
\frac{d}{d x}\left(\frac{x^{n+1}}{n+1}\right)=x^{n}, \text { so } \int x^{n} d x=\frac{x^{n+1}}{n+1}+c \text { for } n \neq-1
$$

In words, to integrate the function $x^{n}$, we add 1 to the power (i.e., $n+1$ ) and divide the function by $n+1$. If $n=1$, we have a special case

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Rule 2: Trigonometric functions.

$$
\begin{gathered}
\frac{d}{d x} \sin x=\cos x, \text { so } \int \cos x d x=\sin x+c \\
\frac{d}{d x} \cos x=-\sin x, \text { so } \int-\sin x d x=\cos x+c
\end{gathered}
$$

Therefore, $\int \sin x d x=-\cos x+c$.

The other trigonometric functions with the previous rules are listed in the following table:

| Derivative | Indefinite Integral |
| :--- | :--- |
| $\frac{d}{d x}(x)=1$ | $\int 1 d x=x+c$ |
| $\frac{d}{d x}\left(\frac{x^{n+1}}{n+1}\right)=x^{n}, n \neq-1$ | $\int x^{n} d x=\frac{x^{n+1}}{n+1}+c$ |
| $\frac{d}{d x}(\sin x)=\cos x$ | $\int \cos x d x=\sin x+c$ |
| $\frac{d}{d x}(-\cos x)=\sin x$ | $\int \sin x d x=-\cos x+c$ |
| $\frac{d}{d x}(\tan x)=\sec ^{2} x$ | $\int \sec ^{2} x d x=\tan x+c$ |
| $\frac{d}{d x}(-\cot x)=\csc ^{2} x$ | $\int \csc ^{2} x d x=-\cot x+c$ |
| $\frac{d}{d x}(\sec x)=\sec x \tan x$ | $\int \sec x \tan x d x=\sec x+c$ |
| $\frac{d}{d x}(-\csc x)=\csc x \cot x$ | $\int \csc x \cot x d x=-\csc x+c$ |

## Example

Evaluate the integral.
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Solution:
(1) $\int x^{-3} d x=\frac{x^{-2}}{-2}+c=-\frac{1}{2 x^{2}}+c$.
(2) $\int \frac{1}{\cos ^{2} x} d x=\int \sec ^{2} x d x=\tan x+c$.
$\left(\sec x=\frac{1}{\cos x} \Rightarrow \sec ^{2} x=\frac{1}{\cos ^{2} x}\right)$

## Properties of the Indefinite Integral

## Theorem

Assume $f$ and $g$ have antiderivatives on an interval I, then
(1) $\frac{d}{d x} \int f(x) d x=f(x)$.
(2) $\int \frac{d}{d x}(F(x)) d x=F(x)+c$.
(3) $\int(f(x) \pm g(x)) d x=\int f(x) d x \pm \int g(x) d x$.
(4) $\int k f(x) d x=k \int f(x) d x$, where $k$ is a constant.

## Example

Evaluate the integral.
(1) $\int(4 x+3) d x$
(2) $\int(2 \sin x+3 \cos x) d x$
(9) $\int \frac{d}{d x}(\sin x) d x$
(3) $\int\left(\sqrt{x}+\sec ^{2} x\right) d x$
(6) $\frac{d}{d x} \int \sqrt{x+1} d x$

## Solution:

(1) $\int(4 x+3) d x=\frac{4 x^{2}}{2}+3 x+c=2 x^{2}+3 x+c$.
(2) $\int(2 \sin x+3 \cos x) d x=-2 \cos x+3 \sin x+c$.
(3) $\int\left(\sqrt{x}+\sec ^{2} x\right) d x=\frac{x^{\frac{3}{2}}}{3 / 2}+\tan x+c=\frac{2 x^{\frac{3}{2}}}{3}+\tan x+c$.
(4) $\int \frac{d}{d x}(\sin x) d x=\sin x+c$.
(5) $\frac{d}{d x} \int \sqrt{x+1} d x=\sqrt{x+1}$.

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## Example

If $\int f(x) d x=x^{2}+c_{1}$ and $\int g(x) d x=\tan x+c_{2}$, find $\int(3 f(x)-2 g(x)) d x$.

## Solution:

(1) $\int(4 x+3) d x=\frac{4 x^{2}}{2}+3 x+c=2 x^{2}+3 x+c$.
(2) $\int(2 \sin x+3 \cos x) d x=-2 \cos x+3 \sin x+c$.
(3) $\int\left(\sqrt{x}+\sec ^{2} x\right) d x=\frac{x^{\frac{3}{2}}}{3 / 2}+\tan x+c=\frac{2 x^{\frac{3}{2}}}{3}+\tan x+c$.
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## Example

If $\int f(x) d x=x^{2}+c_{1}$ and $\int g(x) d x=\tan x+c_{2}$, find $\int(3 f(x)-2 g(x)) d x$.

## Solution:

From the third and fourth properties,
$\int_{c=3 c_{1}-2 c_{2}}(3 f(x)-2 g(x)) d x=3 \int f(x) d x-2 \int g(x) d x=3 x^{2}-2 \tan x+c$, where

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Solution:

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\begin{aligned}
\int f^{\prime}(x) d x & =\int x^{3} d x \\
\Rightarrow f(x) & =\frac{1}{4} x^{4}+c
\end{aligned}
$$

If $x=0$, then $f(0)=\frac{1}{4}(0)^{4}+c=1$ and this implies $c=1$. Hence, the solution of the differential equation is $f(x)=\frac{1}{4} x^{4}+1$.

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## Notes:

- When evaluating integrals, we can always check our answers by deriving the results.
$\square$ In the previous examples, we use $x$ as a variable of the integration. However, for this role, we can use any variable such as $y, z, t$, etc. That is, instead of $f(x) d x$, we can integrate $f(y) d y$ or $f(t) d t$.


## Integration By Substitution

## Theorem

Let $g$ be a differentiable function on an interval I where the derivative is continuous. Let $f$ be continuous on the interval $J$ contains the range of the function $g$. If $F$ is an antiderivative of the function $f$ on $J$, then

$$
\int f(g(x)) g^{\prime}(x) d x=F(g(x))+c, \quad x \in I
$$

## Example

Evaluate the integral $\int 2 x\left(x^{2}+1\right)^{3} d x$

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## Example

Evaluate the integral $\int 2 x\left(x^{2}+1\right)^{3} d x$

## Solution:

We can use the previous theorem as follows:
let $f(x)=x^{3}$ and $g(x)=x^{2}+1$, then $f(g(x))=\left(x^{2}+1\right)^{3}$. Since $g^{\prime}(x)=2 x$, then from the theorem, we have

$$
\int 2 x\left(x^{2}+1\right)^{3} d x=\frac{\left(x^{2}+1\right)^{4}}{4}+c
$$

We can end with the same solution by using the five steps of the substitution method given below.
$\square$ Steps of the integration by substitution:
Step 1: Choose a new variable $u$.
Step 2: Determine the value of $d u$.
Step 3: Make the substitution i.e., eliminate all occurrences of $x$ in the integral by making the entire integral in terms of $u$.
Step 4: Evaluate the new integral.
Step 5: Return the evaluation to the initial variable $x$.

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Step 5: Return the evaluation to the initial variable $x$.
In the previous example, let $u=x^{2}+1$, then $d u=2 x d x$. By substituting that into the original integral, we have

$$
\int u^{3} d u=\frac{u^{4}}{4}+c
$$

Now, by returning the evaluation to the initial variable $x$, we have $\int 2 x\left(x^{2}+1\right)^{3} d x=\frac{\left(x^{2}+1\right)^{4}}{4}+c$.

## Example

Evaluate the integral $\int \frac{\sec ^{2} \sqrt{x}}{\sqrt{x}} d x$.

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## Solution:

We use the theorem for the integral $2 \int \frac{\sec ^{2} \sqrt{x}}{2 \sqrt{x}} d x$. Let $f(x)=\sec ^{2} x$ and $g(x)=\sqrt{x}$, then $f(g(x))=\sec ^{2} \sqrt{x}$. Since $g^{\prime}(x)=1 /(2 \sqrt{x})$, then we have

$$
\int \frac{\sec ^{2} \sqrt{x}}{\sqrt{x}} d x=2 \tan \sqrt{x}+c
$$

## Example

Evaluate the integral $\int \frac{\sec ^{2} \sqrt{x}}{\sqrt{x}} d x$.

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$$
\int \frac{\sec ^{2} \sqrt{x}}{\sqrt{x}} d x=2 \tan \sqrt{x}+c
$$

By using the steps of the substitution method, let $u=\sqrt{x}$, then $d u=\frac{1}{2 \sqrt{x}} d x$. By substitution, we obtain

$$
2 \int \sec ^{2} u d u=2 \tan u+c=2 \tan \sqrt{x}+c
$$

## Example

Evaluate the integral $\int \frac{x^{2}-1}{\left(x^{3}-3 x+1\right)^{6}} d x$.

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Solution:
Let $u=x^{3}-3 x+1$, then $d u=3\left(x^{2}-1\right) d x$. By substitution, we have

$$
\frac{1}{3} \int u^{-6} d u=\frac{1}{3} \frac{1}{-5 u^{5}}+c=\frac{-1}{15\left(x^{3}-3 x+1\right)^{5}}+c
$$

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$$

The upcoming corollary simplifies the process of the substitution method for some functions.

## Corollary

If $\int f(x) d x=F(x)+c$, then for any $a \neq 0$,

$$
\int f(a x \pm b) d x=\frac{1}{a} F(a x \pm b)+c
$$

## Example

Evaluate the integral.
(1) $\int \sqrt{2 x-5} d x$
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## Solution:

From the corollary, we have
(1) $\int \sqrt{2 x-5} d x=\frac{1}{2} \frac{(2 x-5)^{3 / 2}}{3 / 2}+c=\frac{(2 x-5)^{3 / 2}}{3}+c$.

## Example

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(1) $\int \sqrt{2 x-5} d x$
(2) $\int \cos (3 x+4) d x$

## Solution:

From the corollary, we have
(1) $\int \sqrt{2 x-5} d x=\frac{1}{2} \frac{(2 x-5)^{3 / 2}}{3 / 2}+c=\frac{(2 x-5)^{3 / 2}}{3}+c$.
(2) $\int \cos (3 x+4) d x=\frac{1}{3} \sin (3 x+4)+c$.

