Integral Calculus

Department of Mathematics

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Main Contents

1. Natural logarithmic function.
2. Natural exponential function.
3. General exponential and logarithmic function.
4. Main properties of exponential and logarithmic functions.
5. Differentiation and integration of exponential and logarithmic functions.

The Natural Logarithmic Function

In chapter 1, we found that $\int x^r \, dx = \frac{x^{r+1}}{r+1} + c$. If $r = -1$, does the previous rule hold? The answer is no because the denominator will become zero. The task now is to find a general antiderivative of the function $\frac{1}{x}$; meaning that we are looking for a function $F(x)$ such that $F^{'}(x) = \frac{1}{x}$.

The Natural Logarithmic Function

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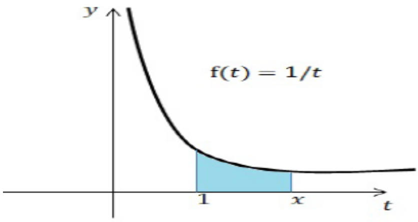
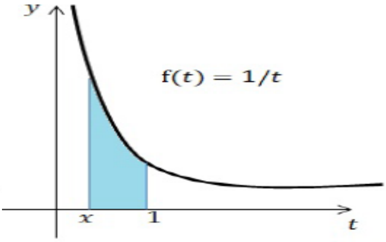
Consider the function $f(t) = \frac{1}{t}$. It is continuous on the interval $(0, +\infty)$ and this implies that the function is integrable on the interval $[1, x]$. Figure 3 shows the graph of the function $f(t) = \frac{1}{t}$ from $t = 1$ to $t = x$ where $x > 0$. The area of the region under the graph can be expressed as

$$f(x) = \int_1^x \frac{1}{t} \, dx$$





Definition

The natural logarithmic function is defined as follows:

$$\ln : (0, \infty) \to \mathbb{R} \ ,$$

$$\ln x = \int_1^x \frac{1}{t} \ dt$$

for every $x > 0$.

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Properties of the Natural Logarithmic Function

1) From the Definition .1, the domain of the function $\ln x$ is $(0, \infty)$.

2) The range of the function $\ln x$ is $\mathbb{R}$ as follows:

$$y = \begin{cases} \ln x > 0 & : x > 1 \\ \ln x = 0 & : x = 1 \\ \ln x < 0 & : 0 < x < 1 \end{cases}$$

To see this, let $x = 1$, then $\ln x = \int_1^1 \frac{1}{t} \ dt = 0$. Now, since $\int_1^x \frac{1}{t} \ dt = -\int_x^1 \frac{1}{t} \ dt$, then for $0 < x < 1$, the integral is the negative of the area of the region under $f(t) = \frac{1}{t}$ from $t = x$ to $x = 1$. This means that $\ln x$ is negative for $0 < x < 1$ and positive for $x > 1$.

3) The function $\ln x$ is differentiable and continuous on the domain. From the fundamental theorem of calculus, we have

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}\int_1^x \frac{1}{t}\, dt = \frac{1}{x}, \forall x > 0.$$

Therefore, the function $\ln x$ is increasing on the interval $(0, \infty)$.
4) The second derivative $\frac{d^2}{dx^2}(\ln x) = \frac{-1}{x^2} < 0$ for all $x \in (0, \infty)$. Therefore, the function $\ln x$ is concave downward on the interval $(0, \infty)$.
5) Rules of the natural logarithmic function:

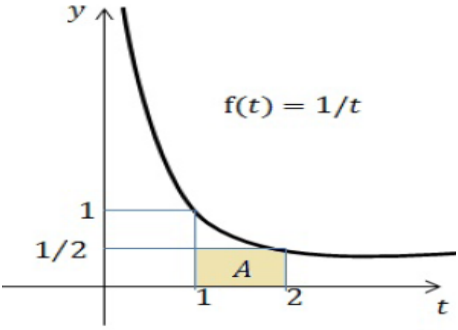
Theorem

If $a, b > 0$ and $r \in \mathbb{Q}$, then

1. $\ln ab = \ln a + \ln b$.
2. $\ln \frac{a}{b} = \ln a - \ln b$.
3. $\ln a^r = r \ln a$.

6) $\lim\limits_{x\to\infty} \ln x = \infty$ and $\lim\limits_{x\to 0^+} \ln x = -\infty$.

To see this, the figure on the right shows the region of $f(t) = \frac{1}{t}$ from $t = 1$ to $t = x$. The area $A = (1)(\frac{1}{2}) = \frac{1}{2}$. From Definition .1, $\ln 2 = \displaystyle\int_1^2 \frac{1}{t}\, dt > \frac{1}{2} =$ area of A.



Since $\ln x$ is increasing function, then

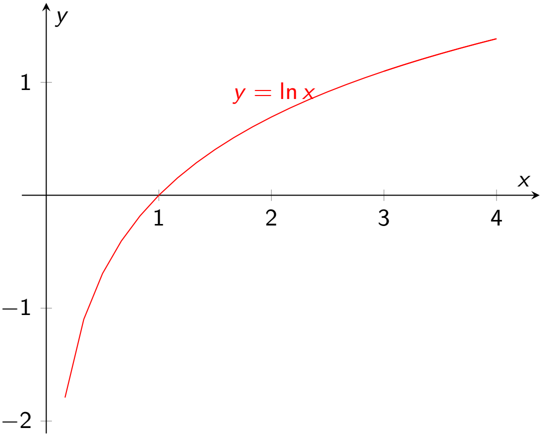
$$\ln x > \ln 2^m = m \ln 2 > \frac{m}{2} \ \forall m \in \mathbb{N}$$

where if m is sufficiently large, $x \geq 2^m$. This implies $\lim\limits_{x\to\infty} \ln x > \frac{m}{2}$, then $\lim\limits_{x\to\infty} \ln x = \infty$.

Now, let $u = \frac{1}{x}$ as $x \to 0^+$, $u \to \infty$. Since $x = \frac{1}{u} \Rightarrow \ln x = \ln \frac{1}{u} = -\ln u$. This implies

$$\lim_{x\to 0^+} \ln x = \lim_{x\to\infty} (-\ln u) = -\lim_{x\to\infty} \ln u = -\infty.$$

From the previous properties, we have the graph of the function $y = \ln x$.



The graph of the function $y = \ln x$.

We have found that

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Hence,

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}\ln(|\, x \,|) = \frac{1}{x} \;\forall x \neq 0.$$

Theorem

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\ln u = \frac{1}{u}\, u'$$

Example

Find the derivative of the function.

1. $f(x) = \ln(x+1)$
2. $g(x) = \ln(x^3 + 2x - 1)$
3. $h(x) = \ln\sqrt{x^2+1}$
4. $y(x) = \sqrt{\ln x}$
5. $f(x) = \ln\cos\ x$
6. $g(x) = \sqrt{x}\ \ln x$
7. $h(x) = \sin\ (\ln x)$
8. $y(x) = \ln(x + \ln x)$

Solution:

1. $f'(x) = \frac{1}{x+1}$.
2. $g'(x) = \frac{3x^2+2}{x^3+2x-1}$.
3. $h'(x) = \frac{1}{\sqrt{x^2+1}} \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{x^2+1}$.
4. $y'(x) = \frac{1}{2\sqrt{\ln x}} \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$.
5. $f'(x) = \frac{-\sin\ x}{\cos\ x} = -\tan\ x$.
6. $g'(x) = \frac{1}{2\sqrt{x}} \ln x + \sqrt{x}\frac{1}{x} = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} = \frac{\ln x+2}{2\sqrt{x}}$.
7. $h'(x) = \cos\ (\ln x)(\frac{1}{x}) = \frac{\cos\ (\ln x)}{x}$.
8. $y'(x) = \frac{1}{x+\ln x}(1 + \frac{1}{x}) = \frac{x+1}{x(x+\ln x)}$.

Example

Find the derivative of the function $y = \sqrt[5]{\frac{x-1}{x+1}}$.

Solution:

We can solve this example using the derivative rules. However, for simplicity, we use the natural logarithmic function.

By Taking the logarithm function of each side, we have

$$\ln \mid y \mid = \ln \left| \sqrt[5]{\frac{x-1}{x+1}} \right| = \frac{1}{5}\Big(\ln|x-1| - \ln|x+1| \Big).$$

By differentiating both sides with respect to x, we have

$$\frac{y'}{y} = \frac{1}{5}\Big(\frac{1}{x-1} - \frac{1}{x+1}\Big) \qquad \Big(\frac{d}{dx}\ln y = \frac{y'}{y}\Big)$$

By multiplying both sides by y, we obtain

$$y' = \frac{1}{5}\Big(\frac{1}{x-1} - \frac{1}{x+1}\Big)y$$

$$\Rightarrow y' = \frac{1}{5}\Big(\frac{1}{x-1} - \frac{1}{x+1}\Big)\sqrt[5]{\frac{x-1}{x+1}}.$$

Recall, $\frac{d}{dx} \ln \mid u \mid = \frac{u'}{u}$ where $u = g(x)$ is a differentiable function. By integrating both sides, we have

$$\int \frac{u'}{u} \; dx = \int \frac{d}{dx} \ln \mid u \mid \; dx$$
$$= \ln \mid u \mid + c.$$

This can be stated as follows:

$$\boxed{\int \frac{u'}{u} \; dx \; = \; \ln \mid u \mid + c}$$

If $u = x$, we have the following special case

$$\boxed{\int \frac{1}{x} \; dx \; = \; \ln \mid x \mid + c}$$

Example

Evaluate the integral.

1. $\displaystyle\int \frac{2x}{x^2+1}\,dx$
2. $\displaystyle\int \frac{6x^2+1}{4x^3+2x+1}\,dx$
3. $\displaystyle\int_2^e \frac{dx}{x\,\ln x}$
4. $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}\,(1+\sqrt{x})}$
5. $\displaystyle\int \tan\,x\,dx$
6. $\displaystyle\int \cot\,x\,dx$
7. $\displaystyle\int \sec\,x\,dx$
8. $\displaystyle\int \csc\,x\,dx$

Solution:

1) $\displaystyle\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1)+c.$

Example

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7. $\displaystyle\int \sec\ x\ dx$
8. $\displaystyle\int \csc\ x\ dx$

Solution:

1) $\displaystyle\int \frac{2x}{x^2+1}\, dx = \ln(x^2+1) + c.$

2) $\displaystyle\int \frac{6x^2+1}{4x^3+2x+1}\, dx = \frac{1}{2}\int \frac{12x^2+2}{4x^3+2x+1}\, dx = \frac{1}{2}\ln \mid 4x^3+2x+1 \mid + c.$

Example

Evaluate the integral.

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2. $\displaystyle\int \frac{6x^2+1}{4x^3+2x+1}\ dx$
3. $\displaystyle\int_2^e \frac{dx}{x\ \ln x}$
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Solution:

1) $\displaystyle\int \frac{2x}{x^2+1}\ dx = \ln(x^2+1)+c.$

2) $\displaystyle\int \frac{6x^2+1}{4x^3+2x+1}\ dx = \tfrac{1}{2}\int \frac{12x^2+2}{4x^3+2x+1}\ dx = \tfrac{1}{2}\ln\mid 4x^3+2x+1\mid +c.$

3) Let $u=\ln x$, then $du=\frac{1}{x}\ dx$. By substitution, we obtain $\displaystyle\int \frac{1}{u}\ du = \ln\mid u\mid.$

By returning the evaluation to the initial variable x, we have $\displaystyle\int \frac{dx}{x\ \ln x} = \ln(\ln x).$

Hence,

$$\int_2^e \frac{dx}{x\ \ln x} = \Big[\ln(\ln x)\Big]_2^e = \ln(\ln e) - \ln(\ln 2) = \ln(1) - \ln(\ln 2) = -\ln(\ln 2).$$

4) For $\displaystyle\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})}$, let $u = 1+\sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}\,dx$. By substitution, we have $\displaystyle 2\int \frac{1}{u}\,du = 2\ln \mid u \mid$.

By returning the evaluation to the initial variable x, we have $\displaystyle\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = \ln \mid 1+\sqrt{x} \mid$. Hence,

$$\int_1^4 \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = 2\Big[\ln \mid 1+\sqrt{x} \mid\Big]_1^4 = 2(\ln 3 - \ln 2).$$

4) For $\displaystyle\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})}$, let $u = 1+\sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}\ dx$. By substitution, we have $\displaystyle 2\int \frac{1}{u}\ du = 2\ln \mid u \mid$.
By returning the evaluation to the initial variable x, we have $\displaystyle\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = \ln \mid 1+\sqrt{x} \mid$. Hence,

$$\int_1^4 \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = 2\Big[\ln \mid 1+\sqrt{x} \mid\Big]_1^4 = 2(\ln 3 - \ln 2).$$

5) We know that $\tan\ x = \frac{\sin\ x}{\cos\ x}$. Therefore,

$$\int \tan\ x\ dx = \int \frac{\sin\ x}{\cos\ x}\ dx = -\int \frac{-\sin\ x}{\cos\ x}\ dx$$
$$= -\ln \mid \cos\ x \mid + c = \ln \mid \sec\ x \mid + c.$$

$(\sec\ x = \frac{1}{\cos\ x})$

6) $\int \cot \ x \ dx = \int \frac{\cos \ x}{\sin \ x} \ dx = \ln \mid \sin \ x \mid + c = -\ln \mid \csc \ x \mid + c$

$(\csc \ x = \frac{1}{\sin \ x})$

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$(\csc\ x = \frac{1}{\sin\ x})$

7)

$$\begin{aligned}\int \sec\ x\ dx &= \int \frac{\sec\ x\ (\sec\ x + \tan\ x)}{(\sec\ x + \tan\ x)}\ dx \\ &= \int \frac{\sec^2\ x + \sec\ x \tan x}{\sec\ x + \tan\ x}\ dx \\ &= \ln|\sec\ x + \tan\ x| + c\ .\end{aligned}$$

6) $\int \cot\ x\ dx = \int \frac{\cos\ x}{\sin\ x}\ dx = \ln|\sin\ x| + c = -\ln|\csc\ x| + c$
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8)

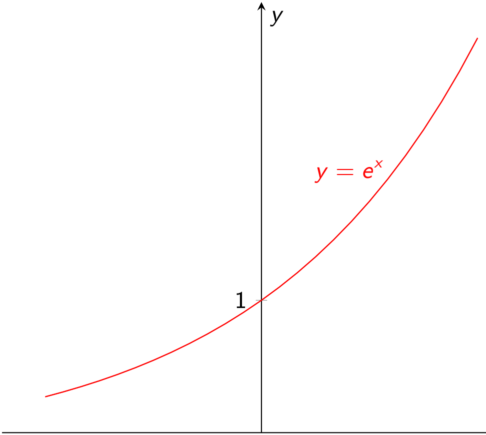
$$\begin{aligned}\int \csc\ x\ dx &= \int \frac{\csc\ x\ (\csc\ x - \cot\ x)}{(\csc\ x - \cot\ x)}\ dx \\ &= \int \frac{\csc^2\ x - \csc\ x\ \cot\ x}{\csc\ x - \cot\ x}\ dx \\ &= \ln|\csc\ x - \cot\ x| + c\ .\end{aligned}$$

Definition

The natural exponential function is defined as follows:

$$exp : \mathbb{R} \longrightarrow (0, \infty) \ ,$$

$$y = exp\ x \Leftrightarrow \ln y = x$$



figureThe graph of the function $y = e^x$.

Properties of the Natural Exponential Function

1) From the definition, the domain of the function $exp\ x$ is $\mathbb{R}$.

2) The range of the function $exp\ x$ is $(0, \infty)$ as follows:

$$y = \begin{cases} exp\ x > 1 & : x > 0 \\ exp\ x = 1 & : x = 0 \\ exp\ x < 1 & : x < 0 \end{cases}$$

3) Usually, the symbol *exp* x is written as e^x, so *exp* $(1) = e \approx 2.71828$. From Definition .2, we have $\ln e = 1$ and $\ln e^r = r \ln e = r \; \forall \; r \in \mathbb{Q}$.
4) The function e^x is continuous and differentiable on the domain. From Definition .2, we have

$$y = e^x \Rightarrow \ln y = x.$$

By differentiating both sides, we have

$$\frac{d}{dx} \ln y = \frac{y'}{y} = 1 \Rightarrow y' = y.$$

Hence,

$$\frac{d}{dx} e^x = e^x \; \forall x \in \mathbb{R}.$$

Therefore, the function e^x is increasing on the domain $\mathbb{R}$.
5) The second derivative $\frac{d^2}{dx^2} e^x = e^x > 0$ for all $x \in \mathbb{R}$. Hence, the function e^x is concave upward on the domain $\mathbb{R}$.
6) $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to -\infty} e^x = 0$.
7) Since e^x and $\ln x$ are inverse functions, then

$$\ln e^x = x, \;\; \forall x \in \mathbb{R} \;,$$

$$e^{\ln x} = x, \;\; \forall x \in (0, \infty).$$

8) Rules of the natural exponential function:

Theorem

If $a, b > 0$ *and* $r \in \mathbb{Q}$, *then*

1. $e^a e^b = e^{a+b}$
2. $\dfrac{e^a}{e^b} = e^{a-b}$
3. $(e^a)^r = e^{ar}$

Example

Solve for x.

1. $\ln x = 2$
2. $\ln(\ln x) = 0$
3. $(x-1)e^{-\ln\frac{1}{x}} = 2$
4. $xe^{2\ln x} = 8$

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1) $\ln x = 2 \Rightarrow e^{\ln x} = e^2 \Rightarrow x = e^2$.

(take *exp* of both sides)

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(take *exp* of both sides)

2) $\ln(\ln x) = 0 \Rightarrow e^{\ln(\ln x)} = e^0 \Rightarrow \ln x = 1 \Rightarrow e^{\ln x} = e^1 \Rightarrow x = e$.
(take *exp* twice)

3) $(x-1)e^{-\ln\frac{1}{x}} = 2 \Rightarrow (x-1)e^{\ln(x^{-1})^{-1}} = 2 \Rightarrow (x-1)e^{\ln x} = 2$. This implies

$$x(x-1) = 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1 \text{ or } x = 2.$$

We have to ignore $x = -1$ since the domain of the natural logarithmic function is $(0, \infty)$.

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4) $xe^{2\ln x} = 8 \Rightarrow xe^{\ln x^2} = 8 \Rightarrow x^3 = 8 \Rightarrow x = 2$.

Example

Simplify the expressions.

1. $\ln(e^{\sqrt{x}})$
2. $e^{\frac{1}{3}\ln x}$
3. $(x+1)\ln(e^{x-1})$
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2. $e^{\frac{1}{3}\ln x} = e^{\ln\sqrt[3]{x}} = \sqrt[3]{x}$.

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3) $(x-1)e^{-\ln\frac{1}{x}} = 2 \Rightarrow (x-1)e^{\ln(x^{-1})^{-1}} = 2 \Rightarrow (x-1)e^{\ln x} = 2$. This implies

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Solution:

1. $\ln(e^{\sqrt{x}}) = \sqrt{x}$.
2. $e^{\frac{1}{3}\ln x} = e^{\ln\sqrt[3]{x}} = \sqrt[3]{x}$.
3. $(x+1)\ln(e^{x-1}) = (x+1)(x-1) = x^2 - 1$.
4. $e^{(\sqrt{x}+2\ln x)} = e^{\sqrt{x}}e^{\ln x^2} = x^2 e^{\sqrt{x}}$.

Definition (The general exponential function) :
It has the form a^x where $a > 0$ and $a \neq 1$.
Note : $a^x = e^{x \ln a}$.
Derivative of the general exponential function :

1. $$\frac{d}{dx}a^x = a^x \ln a.$$
2. $$\frac{d}{dx}a^{f(x)} = a^{f(x)} f'(x) \ln a \ .$$

Integration :

1. $\int a^x = \frac{a^x}{\ln a} + c.$
2. $\int a^{f(x)} f'(x) = \frac{a^{f(x)}}{\ln a} + c$.

Definition (The general logarithmic function) : The general logarithmic function of base a where $a > 0$ and $a \neq 1$ is denoted by $\log_a x$ and it is the inverse function of the general exponential function a^x .

Notes :

1. $\log_a x = y \Leftrightarrow a^y = x$.
2. $\log_a x = \frac{\ln x}{\ln a}$.

Notations :

1. $\log x = \log_{10} x$.
2. $\ln x = \log_e x$.

Derivative of the general logarithmic function :

1. $\frac{d}{dx} \log_a |x| = \frac{1}{x} \frac{1}{\ln a}$.
2. $\frac{d}{dx} \log_a |f(x)| = \frac{f'(x)}{f(x)} \frac{1}{\ln a}$.

1. Find the value of x if $\log_2 x = 3$?.
$\log_2 x = 3 \Leftrightarrow x = 2^3 = 8$.

2. Find the value of a if $\log_a 125 = 3$?
$\log_a 125 = 3 \Leftrightarrow 125 = a^3 \Leftrightarrow a = \sqrt[3]{125} = 5$.

3. Find the value of x if $2\log|x| = \log 2 + \log|3x-4|$? .
$2\log|x| = \log 2 + \log|3x-4| \Rightarrow \log x^2 = \log|2(3x-4)|$
$\Rightarrow x^2 = 2(3x-4) \Rightarrow x^2 = 6x - 8 \Rightarrow x^2 - 6x + 8 = 0$
$(x-4)(x-2) = 0 \Rightarrow x = 4 or x = 2$.

4. Find y' if $2x = 4^y$?
Differentiate both sides : $2 = 4^y y' \ln 4 \Rightarrow y' = \dfrac{2}{4^y \ln 4} = \dfrac{2}{2x\ln 4} = \dfrac{1}{x\ln 4}$.
Another way : $2x = 4^y \Rightarrow \ln|2x| = \ln 4^y = y\ln 4 \Rightarrow y = \dfrac{\ln|2x|}{\ln 4}$
Hence $y' = \dfrac{1}{\ln 4}\dfrac{2}{2x} = \dfrac{1}{x\ln 4}$.

1. Find $f'(x)$ if $f(x) = 7^{\sqrt[3]{x}}$?.
$f'(x) = 7^{\sqrt[3]{x}} \frac{1}{3} x^{\frac{-2}{3}} \ln 7$.

2. Find $f'(x)$ if $f(x) = \pi^{3x}$?
$f'(x) = \pi^{3x}(3) \ln \pi = 3\pi^{3x} \ln \pi$.

3. Find y' if $y = (\sin x)^x$?
$y = (\sin x)^x \Rightarrow \ln y = \ln (\sin x)^x = x \ln |\sin x|$
Differentiate both sides : $\dfrac{y'}{y} = \ln|\sin x| + x\dfrac{\cos x}{\sin x} = \ln|\sin x| + x \cot x$
$y' = y \left[\ln|\sin x| + x \cot x\right] = (\sin x)^x \left[\ln|\sin x| + x \cot x\right]$

4. Find y' if $y = \left(1 + x^2\right)^{2x+1}$?
$y = \left(1 + x^2\right)^{2x+1} \Rightarrow \ln y = \ln \left(1 + x^2\right)^{2x+1} = (2x+1)\ln(1+x^2)$
Differentiate both sides : $\frac{y'}{y} = 2\ln(1+x^2) + (2x+1)\frac{2x}{1+x^2}$
$y' = y \left[2\ln(1+x^2) + \frac{2x(2x+1)}{1+x^2}\right] =$
$\left(1 + x^2\right)^{2x+1} \left[2\ln(1+x^2) + \dfrac{2x(2x+1)}{1+x^2}\right]$

1. $\int x^2 6^{x^3} = \frac{1}{3\ln 6} 6^{x^3}(3x^2)\ln 6 = \frac{6^{x^3}}{3\ln 6} + c$.
2. $\int \frac{2^x}{2^x+1} = \frac{1}{\ln 2}\frac{2^x \ln 2}{2^x+1} = \frac{\ln(2^x+1)}{\ln 2} + c$.
3. $\int \frac{3^{-\cot x}}{\sin^2 x} = \frac{1}{\ln 3} 3^{-\cot x}\csc^2 x \ln 3 = \frac{3^{-\cot x}}{\ln 3} + c$
4. $\int 2^{x\ln x}(1+\ln|x|) = \frac{1}{\ln 2} 2^{x\ln x}(1+\ln|x|)\ln 2 = \frac{2^{x\ln x}}{\ln 2} + c$
5. $\int 4^x 5^{4^x} = \frac{1}{\ln 4\ \ \ln 5} 5^{4^x} 4^x \ln 4\ \ \ln 5 = \frac{5^{4^x}}{\ln 4\ \ \ln 5} + c$
6. $\int 3^x(1+\sin 3^x) = (3^x + 3^x \sin 3^x) = 3^x + 3^x \sin 3^x$
   $= \frac{1}{\ln 3}\int 3^x \ln 3 + \frac{1}{\ln 3}\int \sin(3^x)3^x \ln 3 = \frac{3^x}{\ln 3} - \frac{\cos 3^x}{\ln 3} + c$