MATH107 Vectors and Matrices

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Directional Derivatives

If f is a differentiable function of two variable and $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is unit vector, then

$$D_u f(x,y) = [f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}] \cdot [u_1\mathbf{i} + u_2\mathbf{j}] = f_x(x,y)u_1 + f_y(x,y)u_2$$

Gradient

If f is a differentiable function of two variable. The gradient of vector f in (two-or three-dimensions) is the vector function given by

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}.$$

$$\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

Directional Derivatives (Gradient form)

$$D_u f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

Examples

(1) Let $f(x,y)=x^3y^2$. Find the directional derivatives of f at the point $P(-1,2)$ in direction of vector $a=4\mathbf{i}-3\mathbf{j}$

(2) Let $f(x,y)=yz^3-2x^2$. Find gradient of f at the point $P(2,-3,1)$ in direction of vector $a=4\mathbf{i}-3\mathbf{j}$

(3) Find the directional derivatives of $f(x,y,z)=x^2+3yz+4xy$ at the point $P(1,0,-5)$ in direction of vector $a=2\mathbf{i}-3\mathbf{j}+\mathbf{k}$.

Gradient Theorem 1

Let f be a differentiable function of two variable at point $P(x,y)$.
(i) The maximum value of D_uf at $P(x,y)$ is $\|\nabla f(x,y)\|$.
(ii) The maximum rate of increase of $f(x,y)$ at $P(x,y)$ occurs in the direction of $\nabla f(x,y)$.

Gradient Theorem 2

Let f be a differentiable function of two variable at point $P(x, y)$.
(i) The minimum value of $D_u f$ at $P(x, y)$ is $-\|\nabla f(x, y)\|$.
(ii) The minimum rate of increase of $f(x, y)$ at $P(x, y)$ occurs in the direction of $-\nabla f(x, y)$.

Examples

(1) Let $f(x, y) = x^2 + y^2 - 4z$. Find ∇f and find the direction of maximum rate of increase of f at the point $P(2, -1, 1)$.

Equation of Tangent Planes

An equation for tangent plane to the graph of $F(x, y, z) = 0$ at the point $P_0(x_0, y_0, z_0)$ is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_x(x_0, y_0, z_0)(z - z_0) = 0$$

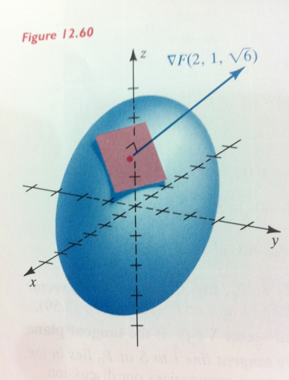

Figure 12.60

Theorem

An equation for tangent plane to the graph of $z = f(x, y) = 0$ at the point $P_0(x_0, y_0, z_0)$ is

$$(z - z_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Equation of Normal line

The line perpendicular to the tangent plane at a point $P_0(x_0, y_0, z_0)$ on a surface S is a normal line to S at P_0. If S is the graph of $F(x, y, z) = 0$, then the normal line is parallel to the vector $\nabla F(x_0, y_0, z_0)$.

Examples

(1) Find an equation for tangent plane to the ellipsoid $\frac{3}{4}x^2 + 3y^2 + z^2 = 12$ at the point $P(2, 1, \sqrt{6})$.

(2) Find an equation for tangent plane to the graph of $x^2 - 4y^2 + z^2 = 16$ at the point $P(2, 1, 4)$.

Examples

(3) Find an equation for tangent plane to the ellipsoid $z = 4x + y^2$ at the point $P(-1, 3, 5)$.

(4) Find an equation of the normal line to the graph of $x^2 - 4y^2 + z^2 = 16$ at the point $P(2, 1, 4)$.

(5) Find equations for the tangent plane and the normal line to the graph of $16x^2 - 9y^2 + 36z^2 = 144$ at the point $P(3, -4, 2)$.

Extrema of function of several variables

Critical point

Let $f(x, y)$ have continuous second derivatives. A point (a, b) is a critical point of f if either
(a) $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or
(b) $f_x(a, b)$ or $f_y(a, b)$ does not exist.

Discriminant D of f

Let $f(x, y)$ be a function of two variables that has continuous second derivatives. The discriminate $D(x, y)$ of $f(x, y)$ is given by:

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2$$

Extrema of function of several variables

Second Derivative Partial Test

Let $f(x, y)$ have continuous second derivatives on an open containing a point (a, b) for which
$f_x(a, b) = 0$ and $f_y(a, b) = 0$
and $D(x, y) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$

1. If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b)
3. If $D < 0$ and $(a, b, f(a, b))$ has a saddle point.
4. Test fails if $D = 0$.

Examples

(1) Find the local extrema and saddle point, if any, of the function $f(x,y) = x^3 + 3xy - y^3$.

(2) Find the local extrema and saddle point, if any, of the function $f(x,y) = \frac{1}{2}x^4 - 2x^3 + 4xy + y^2$.

(3) Find the local extrema and saddle point, if any, of the function $f(x,y) = 3x^2 - 12xy + 4y^3 - 36$.

Lagrange Multipliers

Theorem

Let f and g have continuous first partial derivatives such that f has a local maximum or local minimum at (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x, y) = 0$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

where λ is Lagrange multiplier.

Function of three variables

Suppose $f(x, y, z)$ and $g(x, y, z)$ have continuous first partial derivatives such that f has a local maximum or local minimum at (x_0, y_0, z_0) on the smooth constraint curve $g(x, y, z) = c$. If $\nabla g(x, y, z) = 0$, then there is a real number λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

Lagrange Multipliers of a function subject to more then one constraints

Suppose $f(x, y, z)$ and $g(x, y, z)$ and $h(x, y, z)$ have continuous first partial derivatives such that f has a local maximum or local minimum at (x_0, y_0, z_0) on the smooth constraint curve $g(x, y, z) = c$ and $h(x, y, z) = c$. If $\nabla g(x, y, z) = 0$ and $\nabla g(x, y, z) = 0$, then there is a real number λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

Examples

(1) Use Lagrange multipliers to find maximum value of the function $f(x, y) = xy$ subject to constraint $x + y = 16$.

(2) Use Lagrange multipliers to find greatest and smallest values of the function $f(x, y, z) = x + y + z$ subject to constraint $x^2 + y^2 + z^2 = 25$.

(3) Use Lagrange multipliers to find extrema of the function $f(x, y, z) = z$ subject to constraints $g(x, y, z) = x^2 + y^2 + z^2 - 9$ and $h(x, y, z) = x - y + 3z - 6$.