MATH107 Vectors and Matrices

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Determinant of a Matrix

1- Determinant of a Matrix: Determinant of matrix A is denoted by $|A|$ or $det(A)$.

2- Evaluating determinant by direct multiplication

The determinant of a 2×2 Matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is

$$Det(A) = det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

The determinant of a 3×3 Matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is

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The determinant of a 4×4 Matrix or higher order does not work with this method.

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Example 1: Find determinant of matrices

$$A = \begin{bmatrix} 2 & 1 \\ 7 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

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3- Finding determinant by method of co-factors

Minor: The minor of an element a_{ij} of a matrix A denoted by M_{ij} is determinant of the matrix obtained by deleting the row and column containing a_{ij}. For example $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ M_{23} is the determinant of 2×2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$. Thus

$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$

Cofactor:

Cofactor of an element a_{ij} of a matrix A denoted by C_{ij} is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For example, determinant the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ by method of cofactor.

$Det(A) = a_{11}C_{11} + a_{12}C_{12+a_{13}C_{13}}$ or

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Example 2: Find determinant of matrix using method of co-factor

$$A = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & 1 & 5 \\ 1 & 0 & 4 & 0 \end{bmatrix}$$

Example 3: Find all values of λ for which $det(A) = 0$

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

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3- Evaluating determinant by row operations

1. If matrix A_1 is obtained from matrix A by the interchange of two rows, then $\mathbf{det(A_1) = -det(A)}$.
2. If matrix A_2 is obtained from matrix A by the multiplication of a row, then $\mathbf{det(A_2) = k\,det(A)}$
3. If matrix A_3 is obtained from matrix A by the addition of a multiple of one row to another, then $\mathbf{det(A_3) = det(A)}$

Example 1: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}$$

Find determinant of (i)

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 0 & 1 & 2 \end{bmatrix}$$

(ii)

$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

(iii)

$$A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

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Note:

1. If matrix A is any square matrix that contains a row of zeros,, then $\mathbf{det(A) = 0}$.
2. If a square matrix has two proportional rows, then $\mathbf{det(A) = 0}$.
3. In case of upper or lower triangular matrix, determinant is the product of the diagonal elements

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Example 1: Let

$$A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6$$

Find determinant (i)

$$A_1 = \begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$$

(ii)

$$A_2 = \begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$$

(iii)

$$A_3 = \begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$$

(v)

$$A_4 = \begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

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Theorem

For an $n \times n$ matrix A, following are equivalent

1. $\mathbf{det(A) \neq 0}$.
2. A^{-1} exists
3. If matrix $AX = B$ has a unigue solution for any B.
4. A is invertible