MATH107 Vectors and Matrices

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21/11/16

Equation of a Line

Equation of the line l through the point $P < x_1, y_1, z_1 >$ is parallel to vector $a =< a_1, a_2, a_3 >$ is

Parametric Form

$x = x_1 + a_1 t$

$y = y_1 + a_2 t$

$z = z_1 + a_3 t$, $t \in R$.

Symmetric Form

$\frac{x-x_1}{a_1} = \frac{y-y_1}{a_2} = \frac{z-z_1}{a_3} = t, t \in R$.

Orthogonal and Parallel lines

Let a and b be direction vectors for lines l_1 and l_2,

(1) l_1 and l_2 are orthogonal if $a.b = 0$, and

(2) l_1 and l_2 are parallel if $a = kb$, where k is scalar or

$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = k$.

Examples

(1) Find parametric equation for the line l through $P_1(3,1,-2)$ and $P_2(-2,7,-4)$ passing point $P_1(3,1,-2)$ or $P_2(-2,7,-4)$ and parallel to vector $\overrightarrow{P_1P_2}$

(2) Let lines l_1 and l_2 have respective parametrizations
l_1: $x=-2+3t$, $y=5-4t$, $z=1+2t$, where $t\in\mathbb{R}$
l_2: $x=1-v$, $y=3-2v$, $z=-4-3v$, where $v\in\mathbb{R}$. Find point of intersaction P.

(3) Determine whether the lines
l_1: $x=4-2t$, $y=1+4t$, $z=3+10t$
l_2: $x=v$, $y=6-2v$, $z=1/2-5v$ are parallel.

(4) Determine whether the lines
l_1: $x=-6-t$, $y=10+3t$, $z=3+2t$
l_2: $x=3+2v$, $y=-5-4v$, $z=-1+7v$ are orthogonal.

Equation of a plane

Let $P_1 < x_1, y_1, z_1 >$ lies on the plane and a non-zero vector $a =< a_1, a_2, a_3 >$ is normal to the plane. Let $P < x, y, z >$ be any point in the plane, then $\overrightarrow{P_1P} =< x - x_1, y - y_1, z - z_1 >$. Now

$$a.\overrightarrow{P_1P} = 0$$

$$< a_1, a_2, a_3 > . < x - x_1, y - y_1, z - z_1 >= 0$$

$$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0$$

is equation of the plane.
Linear equation of plane is $ax + by + cz + d = 0$, with the normal vector n is $< a, b, c >$

Notes

Note 1 Distance from a point $P(x_0, y_0, z_0)$ to the plane $ax + by + cz + d = 0$ is

$$h = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Note 2 Shortest distance d between two lines l_1 and l_2 is

$$d = \frac{1}{\|\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}\|} |(\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}).\overrightarrow{P_1P_2}|$$

Note 3 Planes P_1 and P_2 are orthogonal if $n_1.n_2 = 0$.
Note 4 Planes P_1 and P_2 are parallel if $n_1 = kn_2$.

Cylinder

Let C be a curve in a plane, let l be a line that is not in a parallel plane. The set of pionts on all lines that are parallel to l and intersect C is a **cylinder**

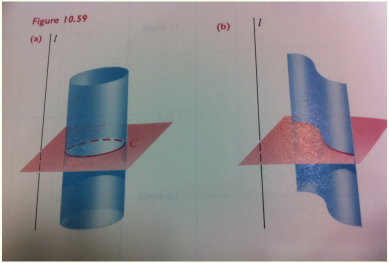
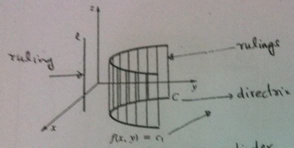
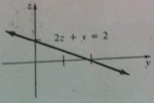


Figure: (a) Right circular cylinder (the directrix is closed curve), (b) Cylinder (the directrix is not closed curve)

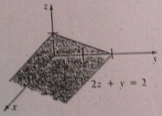


right circular cylinder

Example. 1

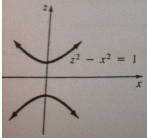


(a) line

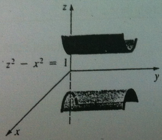


(b) plane

Example. 2

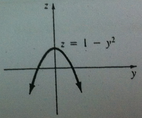


(a) hyperbola

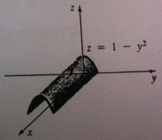


(b) hyperbolic cylinder

Example. 3



(a) parabola

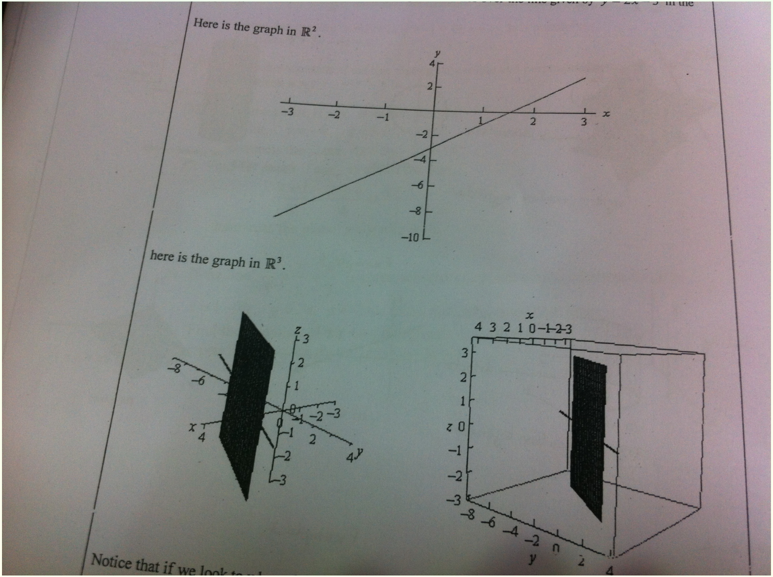


(b) parabolic cylinder

-115-

Example 2: Graph $y = 2x - 3$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example 2: Graph $y = 2x - 3$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.



Here is the graph in $\mathbb{R}^2$.

here is the graph in $\mathbb{R}^3$.

Notice that if we look

Quadric surface

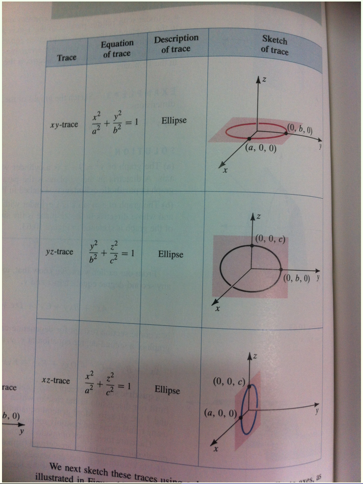
The graph of a second-degree equation in x, y and z

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyx + Gx + Hy + Iz + j = 0$$

is a **quadric surface**.
There are three types of quadric surfaces:
1- ellipsoid
2- hyperboloids.
3- paraboloids.

Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

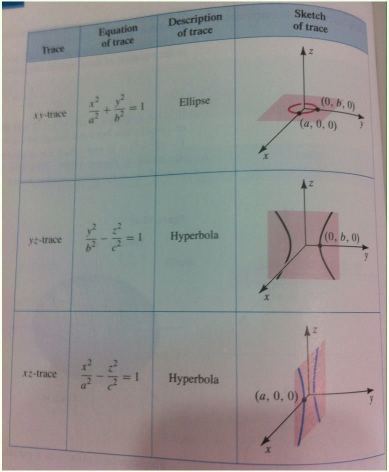


Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Ellipse	$(0, b, 0)$, $(a, 0, 0)$
yz-trace	$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Ellipse	$(0, 0, c)$, $(0, b, 0)$
xz-trace	$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$	Ellipse	$(0, 0, c)$, $(a, 0, 0)$

We next sketch these traces using
illustrated in

Hyperboloid of one Sheet

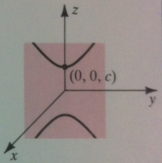
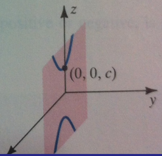
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	Ellipse	z, $(0, b, 0)$, $(a, 0, 0)$, y, x
yz-trace	$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Hyperbola	z, $(0, b, 0)$, y, x
xz-trace	$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$	Hyperbola	z, $(a, 0, 0)$, y, x

Hyperboloid of Two Sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	None	No graph
yz-trace	$-\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Hyperbola	
xz-trace	$-\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$	Hyperbola	

Example 1 Let a surface be $36x^2 + 16y^2 + 9z^2 = 144$.
(1) Write the name of the surface.
(2) write the names and the equation of traces of the surface on the coordinate palnes.
(3) Shetch the surface.
Example 2 Let a surface be $16x^2 - 9y^2 + 36z^2 = 144$.
(1) Write the name of the surface.
(2) write the names and the equation of traces of the surface on the coordinate palnes.
(3) Shetch the surface.
Example 3 Let a surface be $3x^2 - 4y^2 - z^2 = 12$.
(1) Write the name of the surface.
(2) write the names and the equation of traces of the surface on the coordinate palnes.
(3) Shetch the surface.

Example 1

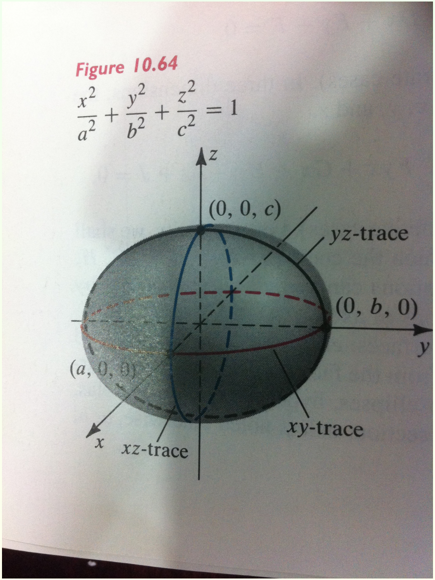


Figure 10.64
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
z
(0, 0, c)
yz-trace
(0, b, 0)
y
(a, 0, 0)
x
xz-trace
xy-trace

Example 2

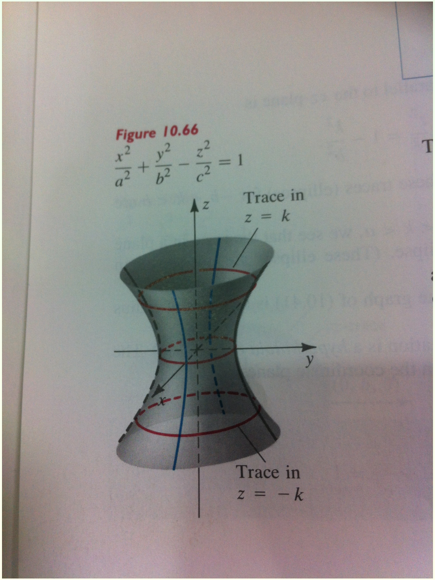


Figure 10.66
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
z
Trace in
z = k
y
x
Trace in
z = −k

Example 3

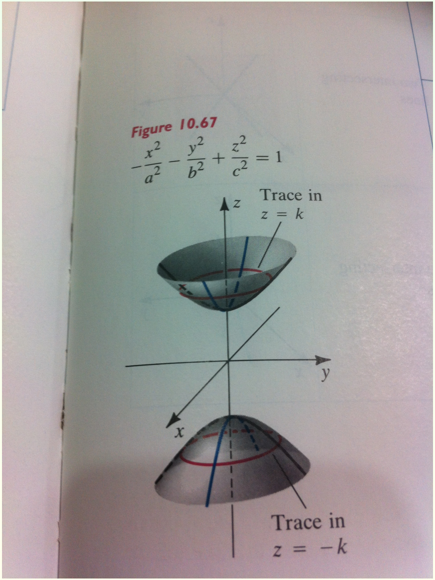
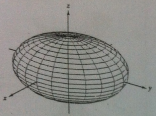


Figure 10.67
$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Trace in
$z = k$
z
y
x
Trace in
$z = -k$

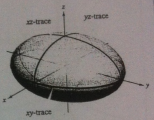


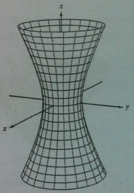
Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Trace	Plane
Ellipse	Parallel to xy-plane
Ellipse	Parallel to xz-plane
Ellipse	Parallel to yz-plane

The surface is a sphere if $a=b=c\neq 0$.



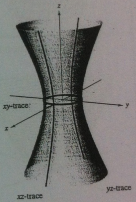


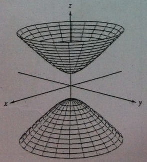
Hyperboloid of One Sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Trace	Plane
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.



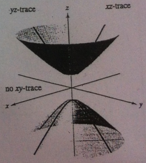


Hyperboloid of Two Sheets

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

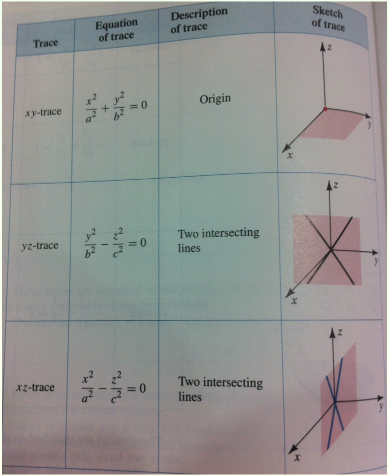
Trace	Plane
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.



Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$



Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$	Origin	
yz-trace	$\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	Two intersecting lines	
xz-trace	$\frac{x^2}{a^2}-\frac{z^2}{c^2}=0$	Two intersecting lines	

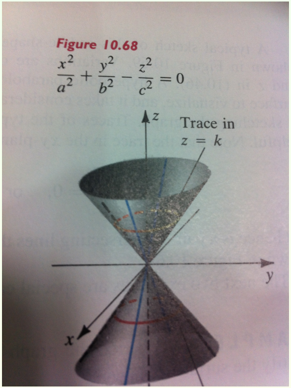
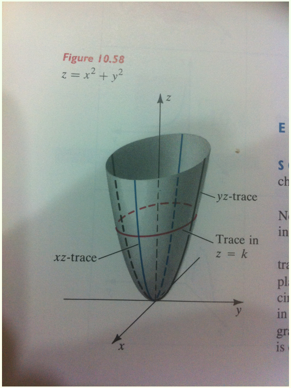


Figure 10.68
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$
z
Trace in
z = k
y
x

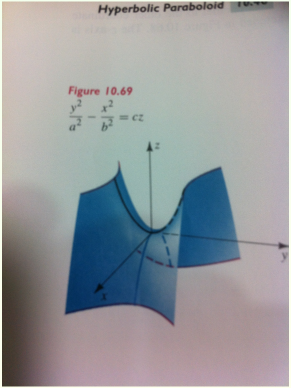
Paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz$$

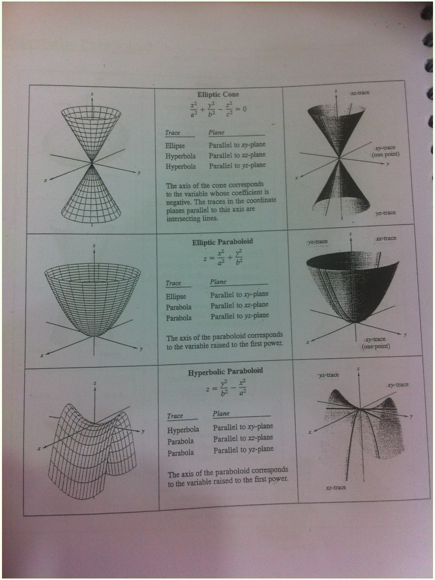


Hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz$$



Example 4 Name of the surface $\frac{x^2}{9}+\frac{y^2}{4}-\frac{z^2}{4}=0$, describe the traces, sketch the graph.
Example 5 Identify $y=x^2+z^2$, sketch the graph.
Example 6 Identify $y=x^2-z^2$, sketch the graph.



Elliptic Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.

Elliptic Paraboloid

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.

Hyperbolic Paraboloid

$$z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$$

Trace	*Plane*
Hyperbola	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.

ALGEBRA

Arithmetic

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$$

Factoring

$$x^2 - y^2 = (x-y)(x+y)$$

$$x^3 - y^3 = (x-y)(x^2+xy+y^2)$$

$$x^3 + y^3 = (x+y)(x^2-xy+y^2)$$

$$x^4 - y^4 = (x-y)(x+y)(x^2+y^2)$$

Binomial

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

Exponents

$$x^n x^m = x^{n+m}$$

$$\frac{x^n}{x^m} = x^{n-m}$$

$$(x^n)^m = x^{nm}$$

$$x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{n/m} = \sqrt[m]{x^n}$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$$

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Lines

Slope m of line through (x_0, y_0) and (x_1, y_1)

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

Through (x_0, y_0), slope m

$$y - y_0 = m(x - x_0)$$

Slope m, y-intercept b

$$y = mx + b$$

Quadratic Formula

If $ax^2 + bx + c = 0$

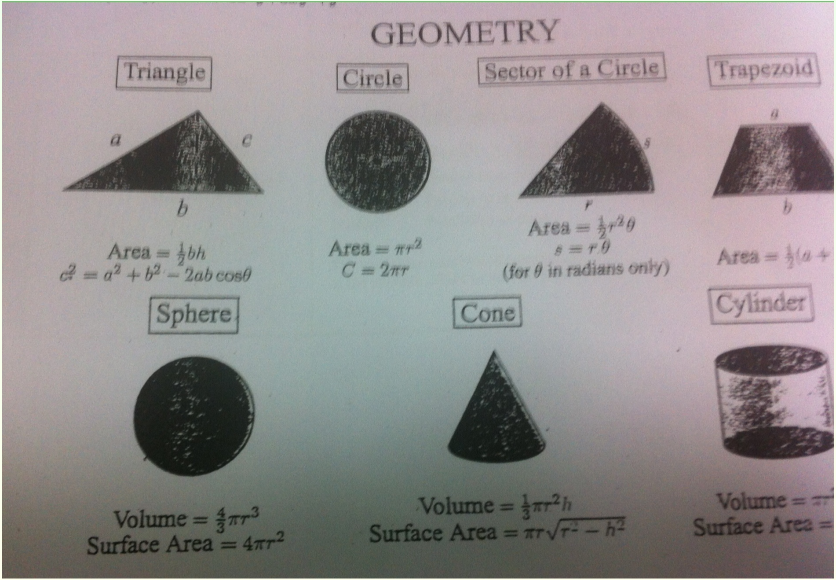
then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Distance

Distance d between (x_1, y_1) and (x_2, y_2)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

GEOMETRY


GEOMETRY
Triangle
Circle
Sector of a Circle
Trapezoid
a
c
b
s
r
a
b
Area $= \frac{1}{2}bh$
$c^2 = a^2 + b^2 - 2ab\cos\theta$
Area $= \pi r^2$
$C = 2\pi r$
Area $= \frac{1}{2}r^2\theta$
$s = r\theta$
(for θ in radians only)
Area $= \frac{1}{2}(a +$
Sphere
Cone
Cylinder
Volume $= \frac{4}{3}\pi r^3$
Surface Area $= 4\pi r^2$
Volume $= \frac{1}{3}\pi r^2 h$
Surface Area $= \pi r\sqrt{r^2 - h^2}$
Volume =
Surface Area =

DERIVATIVE FORMULAS

General Rules

$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$

$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$

$\frac{d}{dx}[cf(x)] = c\,f'(x)$

$\frac{d}{dx}[f(g(x)] = f'(g(x))\,g'(x)$

$\frac{d}{dx}[f(x)\,g(x)] = f'(x)\,g(x) + f(x)\,g'(x)$

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - f(x)g'(x)}{[g(x)]^2}$

Power Rules

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(c) = 0$

$\frac{d}{dx}(cx) = c$

$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

Exponential

$\frac{d}{dx}[e^x] = e^x$

$\frac{d}{dx}[a^x] = a^x \ln a$

$\frac{d}{dx}\left[e^{u(x)}\right] = e^{u(x)}\,u'(x)$

$\frac{d}{dx}[e^{rx}] = r\,e^{rx}$

Trigonometric

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sec x) = \sec x\,\tan x$

$\frac{d}{dx}(\csc x) = -\csc x\,\cot x$

Inverse Trigonometric

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$ $\quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ $\quad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$

Hyperbolic

$\frac{d}{dx}(\sinh x) = \cosh x$ $\quad \frac{d}{dx}(\cosh x) = \sinh x$ $\quad \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$ $\quad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$ $\quad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$ $\quad \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$ $\quad \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$

$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$ $\quad \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$ $\quad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$

other $\frac{d}{dx}(\ln x) = \frac{1}{x}$, $\sinh x = \frac{e^x + e^{-x}}{2}$, $\cosh x = e^{x} -$

-131-

INTEGRAL FORMULAS

General Rules

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx \qquad \int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx$$

$$\int [cf(x)]\, dx = c \int f(x)\, dx \qquad \int [f(x)\, g'(x)]\, dx = f(x)\, g(x) - \int f'(x) g(x)\, dx$$

Power Rules

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1) \qquad \int a\, dx = a\, x + c$$

$$\int \frac{1}{x}\, dx = \ln|x| + c \qquad \int \sqrt{x}\, dx = \tfrac{2}{3} x^{3/2} + c$$

Exponential

$$\int e^x\, dx = e^x + c$$

$$\int a^x\, dx = \frac{1}{\ln a} a^x + c$$

Trigonometric

$$\int \sin x\, dx = -\cos x + c \qquad \int \cos x\, dx = \sin x + c \qquad \int \sec^2 x\, dx = \tan x + c$$

$$\int \csc^2 x\, dx = -\cot x + c \qquad \int \sec x \tan x\, dx = \sec x + c \qquad \int \csc x \cot x\, dx = -\csc x + c$$

$$\int \tan x\, dx = -\ln|\cos x| + c \qquad \int \cot x\, dx = \ln|\sin x| + c \qquad \int \sec x\, dx = \ln|\sec x + \tan x| +$$

Inverse Trigonometric

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c \qquad \int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}x + c$$

Hyperbolic

$$\int \sinh x\,dx = \cosh x + c \qquad \int \cosh x\,dx = \sinh x + c \qquad \int \operatorname{sech}^2 x\,dx = \tanh x + c$$

Inverse Hyperbolic

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1}x + c \qquad \int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}x + c \qquad \int \frac{1}{1-x^2}\,dx = \tanh^{-1}x + c$$