Lines and Planes

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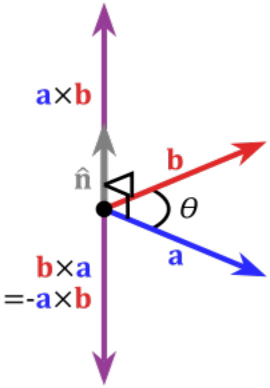
Definition

The *cross product* of two vectors $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$ is the vector

$$a \times b = (\|a\| \,.\, \|b\| \sin(\theta))\, \mathbf{n},$$

where $\mathbf{n}$ is the unit vector of $a \times b$.



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Note That:

$$\begin{array}{ll} \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\ \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{i} = -\mathbf{k} \end{array}$$

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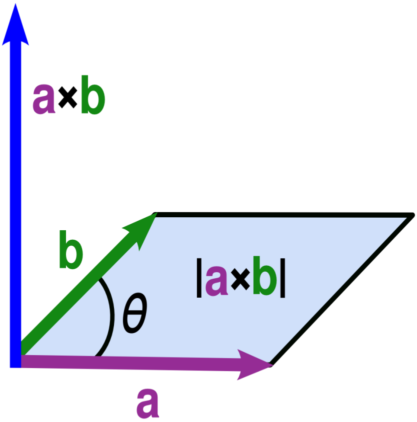

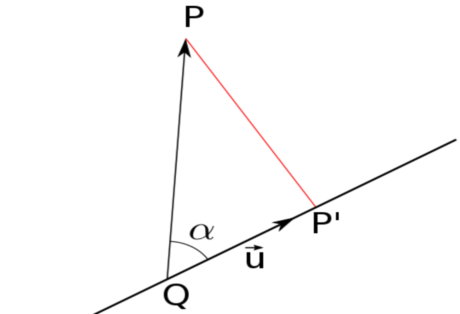
P
α
P'
u
Q

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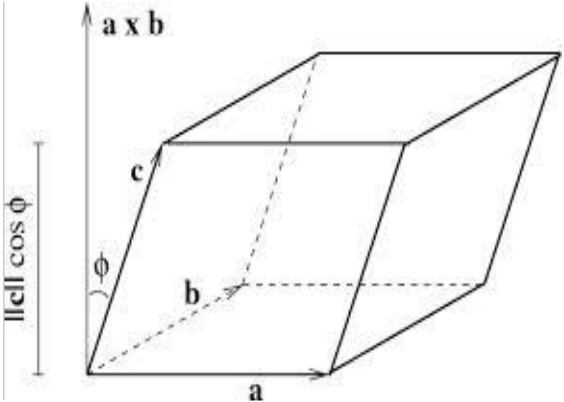
$$d = \|QP\| \sin(\theta) = \frac{\|QP \times QP'\|}{\|QP'\|}$$


a x b
c
||c|| cos φ
φ
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a

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$$v = |(a \times b).c| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & -3 & 2 \\ 3 & -4 & 1 \end{vmatrix} = 4.$$

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Q(3): If ℓ has parametric equations $x = 5 - 3t, y = -2 + t, z = 1 + 9t$, find parametric equation for the line through $P(-6, 4, -3)$ and parallel to ℓ.

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  3. $$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}.$$

Q(1): Determine whether the lines:

$$\begin{aligned} \ell_1 &: \quad x = 4 - 2t, \quad y = 1 + 4t, \quad z = 3 + 10t, \\ \ell_2 &: \quad x = v, \quad y = 6 - 2v, \quad z = \frac{1}{2} - 5v \end{aligned}$$

are parallel.

Q(2): Determine whether the lines:

$$\begin{aligned} \ell_1 &: \quad x = -6 - t, \quad y = 10 + 3t, \quad z = 3 + 2t, \\ \ell_2 &: \quad x = 3 + 2v, \quad y = -5 - 4v, \quad z = -1 + 7v \end{aligned}$$

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intersect, and if so, find the point of intersection.

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Q(4): Find the angle between ℓ_1 and $_2$:

$$\begin{aligned}\ell_1 &: \quad x=4-2t, \quad y=1+4t, \quad z=3+10t,\\ \ell_2 &: \quad x=v, \quad y=6-2v, \quad z==\frac{1}{2}-5v\end{aligned}$$

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