أسئلة سابقة

الفصل الأول 35_36

الجزء الأول:

(1)

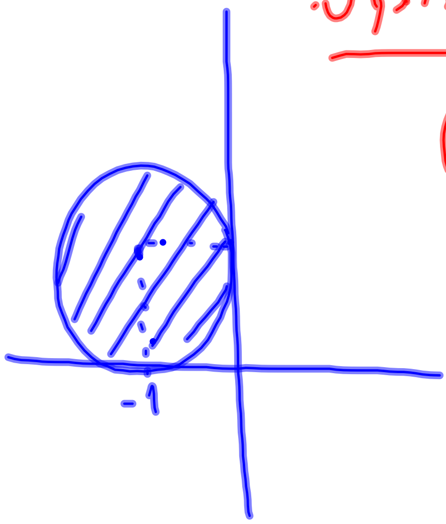
القرص المغلق مركزه $(-1,1)$ ونصف قطره 1 هو مجال للدالة

$$f(x,y)=\sqrt{1-(x+1)^2-(y-1)^2}$$

حيث

$$1-(x+1)^2-(y-1)^2 \geq 0$$

$$\iff (x+1)^2+(y-1)^2 \leq 1 = 1^2$$



(3) الدالة:

$$f(x,y)=\frac{xy^2}{\sqrt{x^2+y^2}}$$

هي دالة متجانسة من الدرجة 2

$$f(tx,ty)=\frac{tx\cdot(ty)^2}{\sqrt{(tx)^2+(ty)^2}}=\frac{t^3\,xy^2}{\sqrt{t^2(x^2+y^2)}}$$

$$=\frac{t^3xy^2}{t\sqrt{x^2+y^2}};\quad t>0$$

$$=t^2\cdot\frac{xy^2}{\sqrt{x^2+y^2}}=t^2f(x,y).$$

(4) مجال الدالة:

$$f(x,y)=\begin{cases}\dfrac{2x+y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$D_f=\mathbb{R}^2$$

(2) نهاية الدالة:

$$f(x,y)=\frac{xy+y^4}{x^2+y^2}$$

عندما (x,y) تقترب من $(0,0)$ هي:

غير موجودة:

$$\lim_{\substack{(0,0)\\x=y}}\frac{xy+y^4}{x^2+y^2}=\lim_{x\to0}\frac{x^2+x^4}{x^2+x^2}=\lim_{x\to0}\frac{x^2(1+x^2)}{2x^2}$$

$$=\lim_{x\to0}\frac{1+x^2}{2}=\frac{1}{2}$$

$$\lim_{\substack{(0,0)\\y=0}}\frac{xy+y^4}{x^2+y^2}=\lim_{x\to0}\frac{0+0^4}{x^2+0^2}=\lim_{x\to0}0=0$$

$$f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

الجزء الثاني

السؤال الأول:

(1) $D_f=\mathbb{R}^2$

(2) لدينا

$$0\le \frac{x^2}{x^2+y^2}\le 1$$

لكل $(x,y)\neq(0,0)$

إذن: $0\le\left|\dfrac{x^3}{x^2+y^2}\right|=\dfrac{x^2}{x^2+y^2}\cdot|x|\le|x|$

ولدينا $\lim\limits_{(0,0)}|x|=0$ ، $\lim\limits_{(0,0)}0=0$

إذن $\lim\limits_{(0,0)}\left|\dfrac{x^3}{x^2+y^2}\right|=0$

وبالتالي: $\lim\limits_{(0,0)}f(x,y)=\lim\limits_{(0,0)}\dfrac{x^3}{x^2+y^2}=0$

(3) عند النقطة (0,0)

لدينا $\lim\limits_{(0,0)}f(x,y)=0=f(0,0)$

إذن: f متصلة عند النقطة $(0,0)$.

عند النقطة $(x,y)\neq(0,0)$

لدينا: $g:(x,y)\longmapsto\dfrac{x^3}{x^2+y^2}$

دالة كسرية وبالتالي g متصلة عند كل نقطة من مجالها: $D_g=\mathbb{R}^2\setminus\{(0,0)\}$

إذن f متصلة عند كل نقطة من $\mathbb{R}^2\setminus\{(0,0)\}$

الاستنتاج: f متصلة عند كل نقطة من مجالها.

$(x,y)\neq(0,0)$ **(5)**

$$\frac{\partial f}{\partial x}(x,y)=\frac{3x^2(x^2+y^2)-x^3\cdot 2x}{(x^2+y^2)^2}=\frac{x^2(x^2+3y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{-x^3\,2y}{(x^2+y^2)^2}=-\frac{2x^3y}{(x^2+y^2)^2}$$

النتيجة:

$$\frac{\partial f}{\partial x}(x,y)=\begin{cases}\dfrac{x^2(x^2+3y^2)}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 1, & (x,y)=(0,0)\end{cases}$$

$$\frac{\partial f}{\partial y}(x,y)=\begin{cases}-\dfrac{2x^3y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

(4)

$$\to \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{\frac{x^3}{x^2}-0}{x}=\lim_{x\to 0}1=1$$

إذن $\boxed{\frac{\partial f}{\partial x}(0,0)=1}$

$$\to \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{\frac{0^3}{y^2}-0}{y}=\lim_{y\to 0}0=0$$

إذن $\boxed{\frac{\partial f}{\partial y}(0,0)=0}$

$$f_{yx}(0,0)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0)=\,?$$

$$\lim_{x\to 0}\frac{\frac{\partial f}{\partial y}(x,0)-\frac{\partial f}{\partial y}(0,0)}{x-0}=\lim_{x\to 0}\frac{0-0}{y}=0$$

إذن $\boxed{f_{yx}(0,0)=0}$

$$f_{xy}(0,0)=\,?\quad\left(f_{xy}(0,0)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0)\right)\ \textbf{(6)}$$

$$\lim_{y\to 0}\frac{\frac{\partial f}{\partial x}(0,y)-\frac{\partial f}{\partial x}(0,0)}{y-0}=\lim_{y\to 0}\frac{0-1}{y}=\mp\infty$$

إذن $f_{xy}(0,0)$ غير موجودة.

السؤال الثاني: (1)

$$w = f(x,y) = \ln(x^2+y^2+1)$$

$$y = r\sin\theta \quad , \quad x = r\cos\theta$$

طريقة 1:

$$w = f(x,y) = \ln\left((r\cos\theta)^2 + (r\sin\theta)^2 + 1\right)$$

$$= \ln\left(r^2(\cos^2\theta + \sin^2\theta) + 1\right) = \ln(r^2+1)$$

$$\boxed{\frac{\partial w}{\partial r} = \frac{2r}{r^2+1} \quad \Big| \quad \frac{\partial w}{\partial \theta} = 0}$$

طريقة 2: (إستعمال قاعدة السلسلة)

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{2x}{x^2+y^2+1}\cdot\cos\theta + \frac{2y}{x^2+y^2+1}\cdot\sin\theta$$

$$\frac{\partial w}{\partial r} = \frac{2r\cos\theta}{r^2+1}\cdot\cos\theta + \frac{2r\sin\theta}{r^2+1}\cdot\sin\theta$$

$$= \frac{2r(\cos^2\theta + \sin^2\theta)}{(r^2+1)} = \boxed{\frac{2r}{r^2+1} = \frac{\partial w}{\partial r}}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{2x}{x^2+y^2+1}\cdot(-r\sin\theta) + \frac{2y}{x^2+y^2+1}\cdot r\cos\theta$$

$$= \frac{2r\cos\theta(-r\sin\theta)}{r^2+1} + \frac{2r\sin\theta \cdot r\cos\theta}{r^2+1}$$

$$= \frac{2r^2(-\cos\theta\sin\theta + \sin\theta\cos\theta)}{r^2+1}$$

$$\boxed{\frac{\partial w}{\partial \theta} = 0}$$

(2.

$$w = f(x,y)$$

$$y = 6u - 4v \quad , \quad x = 3u + 2v$$

$$\frac{\partial^2 w}{\partial u \partial v} = 6\frac{\partial^2 w}{\partial x^2} - 24\frac{\partial^2 w}{\partial y^2}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial w}{\partial x}\cdot 3 + \frac{\partial w}{\partial y}\cdot 6$$

$$\boxed{\frac{\partial w}{\partial u} = 3\frac{\partial w}{\partial x} + 6\frac{\partial w}{\partial y}}$$

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right) = \frac{\partial}{\partial x}\left(3\frac{\partial w}{\partial x} + 6\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(3\frac{\partial w}{\partial x} + 6\frac{\partial w}{\partial y}\right)\frac{\partial y}{\partial v}$$

$$= \left(3\frac{\partial^2 w}{\partial x^2} + 6\frac{\partial^2 w}{\partial y \partial x}\right)2 + \left(3\frac{\partial^2 w}{\partial x \partial y} + 6\frac{\partial^2 w}{\partial y^2}\right)(-4)$$

$$\frac{\partial^2 w}{\partial u \partial v} = 6\frac{\partial^2 w}{\partial x^2} + 12\frac{\partial^2 w}{\partial y \partial x} - 12\frac{\partial^2 w}{\partial x \partial y} - 24\frac{\partial^2 w}{\partial y^2}$$

بما أن f لها مشتقات من الرتبة الثانية متصلة فإن

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}$$

وبالتالي

$$\boxed{\frac{\partial^2 w}{\partial u \partial v} = 6\frac{\partial^2 w}{\partial x^2} - 24\frac{\partial^2 w}{\partial y^2}}$$

3) f دالة في متغيرين متجانسة من الدرجة k.

إذن لكل $t>0$, لكل $(x,y)\in D$ لدينا: $f(tx,ty)=t^{k} f(x,y)$

نشتق المعادلة حسب t:

$$\frac{\partial f}{\partial x}(tx,ty)\cdot\frac{\partial (tx)}{\partial t}+\frac{\partial f}{\partial y}(tx,ty)\cdot\frac{\partial (ty)}{\partial t}=k\,t^{k-1}f(x,y)$$

إذن

$$x\,\frac{\partial f}{\partial x}(tx,ty)+y\,\frac{\partial f}{\partial y}(tx,ty)=k\,t^{k-1}f(x,y)$$

لنأخذ $t=1$ فنحصل على:

$$x\,\frac{\partial f}{\partial x}(x,y)+y\,\frac{\partial f}{\partial y}(x,y)=k\,f(x,y):$$

لكل $(x,y)\in D$