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LOCALLY AND COLOCALLY FACTORABLE BANACH SPACES

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Abstract. We generalize the concept of locality (resp. colocality) to the concept of locally factorable (resp. colocally factorable) such that Theorem 2 of [**2**] and Theorems 1.7 and 1.16 of [**11**] are still valid for the new concepts. In addition we show that locally factorable and colocally factorable are inherited by complemented subspace, then we present some examples and establish relations between locally factorable and colocally factorable. We prove some relations between being finitely (resp. cofinitely) represented in a Banach space and being locally factorable (resp. colocally factorable) some family of finite dimensional Banach spaces.

1. INTRODUCTION

An operator $T : X \longrightarrow Y$ of Banach spaces is an **isomorphism** if it is an invertible bounded linear map, T is an into **isometry** if $\|Tx\| = \|x\|$ for every $x \in X$, it is a **λ-isomorphism,** $\lambda > 1$, if T is an isomorphism and $\|T\| < \lambda$, $\left\|T^{-1}\right\| < \lambda$, Heinrich [**10**, II.6]. **The distance** between two homogeneous maps T_1 and T_2 acting between the same spaces is given by

$$dist\,(T_1, T_2) = \sup\left\{\|T_1x - T_2x\| : \|x\| \leq 1\right\}.$$

We note that bounded maps are those maps at a finite distance from the zero map, also it should be kept in mind that linear maps are not assumed to be bounded. Let $\mathcal{E}$ be a family of finite dimensional Banach spaces, a Banach space X is said to **contain $\mathcal{E}$ uniformly complemented** if there exists a constant c such that for every $E \in \mathcal{E}$, there is a c-complemented subspace A of X which is c- isomorphic to E. It is clear that X contains $\mathcal{E}$ uniformly complemented if and only if its second dual X^{**}does. A Banach space X is said to be λ-**locally $\mathcal{E}$** (or, if no quantitative estimate is needed, **locally $\mathcal{E}$**) if there exists a constant $\lambda > 1$ such that every finite dimensional subspace A of X is contained in a finite dimensional subspace B of X such that $d_{BM}(B, E) < \lambda$, for some $E \in \mathcal{E}$, where $d_{BM}(B, E)$ is the **Banach-Mazur distance** between B and E, and is defined by $d_{BM}(B, E) = \inf\{\|T\|\left\|T^{-1}\right\| : T : B \longrightarrow E$ is an isomorphism of B onto $E\}$. If $\mathcal{E} = \left\{\ell_p^n\right\}_{n=1}^{\infty}$, then X is an **$\mathcal{L}_p$-space,** Lindenstrauss and Rosenthal [**15**].

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A closed subspace Y of a Banach space X is said to be **locally complemented in** X if for every finite dimensional subspace $E \subset X$ there exists an operator $P : E \to Y$ such that P is the identity on $Y \cap E$, with $\|P\| \leq M$ for some M independent of E.

A Banach space X is called **λ-colocally** $\mathcal{E}$ (or **colocally** $\mathcal{E}$) if there exists a constant $\lambda > 1$ such that every finite dimensional quotient A of X is a quotient of another finite dimensional quotient B of X satisfying $d_{BM}(B, E) < \lambda$ for some $E \in \mathcal{E}$, Jebreen, Jamjoom and Yost [**11**]. The space $L_p(\mu)$, for any measure μ, is both locally and colocally $\left\{\ell_p^n\right\}_{n=1}^{\infty}$[**11**, Corollary 1.2].

Let X be a Banach space, $\mathcal{C}_X$ be the set of all finite dimensional subspaces A of X directed by the inclusion, and let $\ell_\infty(A; \mathcal{C}_X)$ be the collection of all $(x_A)_{A\in\mathcal{C}_X} \in \prod_{A\in\mathcal{C}_X} A$ such that $(\|x_A\|)_{A\in\mathcal{C}_X}$ is bounded, with norm given by

$$\left\|(x_A)_{A\in\mathcal{C}_X}\right\|_\infty = \sup_{A\in\mathcal{C}_X} \|x_A\|.$$

Let $\mathcal{U}$ be an ultrafilter on $\mathcal{C}_X$ that refines the corresponding order filter, and let $\left(\prod_{A\in\mathcal{C}_X} A\right)_{\mathcal{U}}$ be the ultraproduct of the family $\mathcal{C}_X$ with respect to the ultrafilter $\mathcal{U}$, that is, $\left(\prod_{A\in\mathcal{C}_X} A\right)_{\mathcal{U}}$ is the quotient space $\ell_\infty(A; \mathcal{C}_X)/N_{\mathcal{U}}$, where $N_{\mathcal{U}} = \left\{(x_A)_{A\in\mathcal{C}_X} \in \ell_\infty(A; \mathcal{C}_X) : \lim_{\mathcal{U}} \|x_A\| = 0\right\}$. The elements of $\left(\prod_{A\in\mathcal{C}_X} A\right)_{\mathcal{U}}$ are denoted by $(x_A)_{\mathcal{U}}$, and its norm is given by $\left\|(x_A)_{\mathcal{U}}\right\| = \lim_{\mathcal{U}} \|x_A\|$, Diestel, Jarchow and Tonge [**8**, p. 170], Sims [**19**, Proposition 4.1, p. 14]. The map $J_X : X \longrightarrow \left(\prod_{A\in\mathcal{C}_X} A\right)_{\mathcal{U}}$ defined by $J_X(x) = (x_A)_{\mathcal{U}}$, where $x_A = x$, if $x \in A$ and $x_A = 0$, otherwise, is an isometry of X onto a subspace of $\left(\prod_{A\in\mathcal{C}_X} A\right)_{\mathcal{U}}$ [**8**, 8.8]. Moreover, the bidual of X is isometrically isomorphic to a quotient of an ultraproduct of the finite dimensional quotient spaces of X [**8**, 8.9].

A diagram $0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{q} Z \longrightarrow 0$ of quasi Banach spaces and bounded linear operators is called **an exact sequence** if the kernel of each arrow coincides with the image of the preceding one. The open mapping theorem implies that X contains $i(Y)$ and the quotient $X/i(Y)$ is isomorphic to Z. In this case, we shall say that X is a **twisted sum** of Y and Z. Two exact sequences $0 \longrightarrow Y \longrightarrow X_1 \longrightarrow Z \longrightarrow 0$ and $0 \longrightarrow Y \longrightarrow X_2 \longrightarrow Z \longrightarrow 0$ are said to be **equivalent** if there is a bounded linear operator T

making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & T\downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

commutative. The three-lemma and the open mapping theorem imply that T must be an isomorphism. An exact sequence $0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$ is said to **split** if it is equivalent to the trivial exact sequence $0 \longrightarrow Y \longrightarrow Y \oplus Z \longrightarrow Z \longrightarrow 0$, in this case, we say that X is **trivial**. We denote by $Ext\,(Z, Y)$ the space of all equivalence classes of locally convex twisted sums of Y and Z. Thus $Ext\,(Z, Y) = 0$ means that all locally convex twisted sums of Y and Z are equivalent to the direct sum $Y \oplus Z$.

A homogeneous map $F : Z \longrightarrow Y$ between two Banach spaces Z and Y is said to be **quasi-linear** if for some constant k and all $z, w \in Z$ it satisfies

$$\|F\,(z + w) - F\,(z) - F\,(w)\| \leq k\,(\|z\| + \|w\|)\,.$$

The smallest constant satisfying the above inequality is called **the quasi-linearity constant** of the map F and is denoted by $Q\,(F)$, Kalton and Peck [**14**]. If $F : Z \longrightarrow Y$ is a quasi-linear map, it is possible to construct a twisted sum $Y \oplus_F Z$ by endowing the product space $Y \times Z$ with the quasi-norm $\|(y, z)\| = \|y - F\,(z)\| + \|z\|$. Clearly, the subspace $\{(y, 0) : y \in Y\}$ of $Y \oplus_F Z$ is isometric to Y and the corresponding quotient $(Y \oplus_F Z)\,/Y$ is isometric to Z . Conversely, given a short exact sequences $0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$, a quasi-linear map $F : Z \longrightarrow Y$ can be obtained such that X is equivalent to $Y \oplus_F Z$, Castillo and González [**4**, 1.5]. Two quasi-linear maps F and G of a Banach space Z into a Banach space Y are said to be **equivalent** if the corresponding exact sequences $0 \longrightarrow Y \longrightarrow Y \oplus_F Z \longrightarrow Z \longrightarrow 0$ and $0 \longrightarrow Y \longrightarrow Y \oplus_G Z \longrightarrow Z \longrightarrow 0$ are equivalent, in this case, we say that F is a **version** of G. It is shown that quasi-linear maps F and G are equivalent if and only if $d\,(F - G, L\,(Z, Y)) = \inf\,\{dist(F - G, L) : L \in L\,(Z, Y)\} < \infty$ [**14,** Theorem 2.5], where $L\,(Z, Y)$ is the space of all linear maps $L : Z \longrightarrow Y$. A quasi-linear map $F : Z \longrightarrow Y$ is said to be **trivial** if the exact sequence $0 \longrightarrow Y \longrightarrow Y \oplus_F Z \longrightarrow Z \longrightarrow 0$ is equivalent to $0 \longrightarrow Y \longrightarrow Y \oplus Z \longrightarrow Z \longrightarrow 0$. Consequently, F is trivial if and only if F is at a finite distance from some linear map, Benyamini and Lindenstrauss [**1**, Theorem 16.2], In particular, F is trivial if and only if it can be written as the sum of a bounded and a linear map. There is a one to one correspondence between the classes of twisted sums $Y \oplus_F Z$ and the classes of quasi-linear maps $F : Z \longrightarrow Y$ [**1,** 16.2]. A homogeneous map $F : Z \longrightarrow Y$ acting between two Banach spaces is said to be **zero-linear** if there is some constant k such that whenever $z_1, z_2, ...z_n$ are finitely many elements of Z then

$$\|F(\Sigma_{i=1}^{n} z_i) - \Sigma_{i=1}^{n} F(z_i)\| \leq k(\Sigma_{i=1}^{n} \|z_i\|).$$

The smallest constant satisfying the above inequality, denoted by $Z(F)$, is called the **zero-linearity constant** of F. We note that a zero-linear map is a quasi-linear map, and that a twisted sum $Y \oplus_F Z$ of Banach spaces Y and Z is locally convex if and only if F is zero-linear, Cabello and Castillo [**3**, Theorem 2] (see also, Castillo and González [**5**, 1.6.e]).

The locality of a family is used to determine the existence of nontrivial twisted sums of certain Banach spaces, in fact, if a Banach space$Y$ is complemented in its bidual Y^{**}, and if $Ext(W, Y) = 0$ for a Banach space W containing a family $\mathcal{E}$ of finite dimensional subspaces uniformly complemented, then $Ext(Z, Y) = 0$ for any Banach space Z which is locally $\mathcal{E}$, Cabello and Castillo [**2**, Theorem 2]. Using this fact, it is shown that there is a nontrivial twisted sums of ℓ_1 and ℓ_2, of ℓ_2 and c_0, and that $Ext(c_0, \ell_1) \neq 0$ [**2**, Examples 4.1, 4.2 and 4.3].

The reader is referred to [**3**, **4**, **5**, **14**] for a detailed account of exact sequences and twisted sums.

2. locally Factorable and colocally Factorable

Let $\mathcal{E}$ be a family of finite dimensional Banach spaces. A Banach space X is said to be **λ-locally $\mathcal{E}-$factorable** (or simply **locally $\mathcal{E}-$factorable**) if there is a constant $\lambda > 1$ such that for every finite dimensional subspace A of X, there is $E_A \in \mathcal{E}$, called **a companion of** A, and there are bounded linear maps φ_A:$A \to E_A$, $\eta_A : E_A \to X$ with $\|\varphi_A\| \leq \lambda$ and $\|\eta_A\| \leq \lambda$ such that $\eta_A \circ \varphi_A = i_A$, where $i_A : A \to X$ is the inclusion map. The maps φ_A, η_A are **the bounded linear factorization** of i_A through E_A, and the diagram

$$\begin{array}{ccc} A & \overset{i_A}{\hookrightarrow} & X \\ \varphi_A \searrow & & \nearrow \eta_A \\ & E_A & \end{array}$$

is called a **locally factorable diagram** for A with respect to E_A. Note that a companion $E_A \in \mathcal{E}$ of A is not unique. It is clear that if a Banach space X is λ-locally $\mathcal{E}$, it is λ-locally $\mathcal{E}-$factorable. Also, it is obvious that if a Banach space X is locally $\mathcal{E}-$factorable, so is every complemented subspace of X and every Banach space isomorphic to X**.** On the other hand, a complemented subspace of a Banach space which is locally $\mathcal{E}$ need not be locally $\mathcal{E}$ also. Indeed, the space $L_p(0,1), 1 < p < \infty$, $p \neq 2$, is locally $\{\ell_p^n\}_{n=1}^{\infty}$, Lindenstrauss and Tzafriri [**16**, II.5.b] and contains a complemented subspace A_p which is not locally $\{\ell_p^n\}_{n=1}^{\infty}$ since it is isomorphic to a Hilbert space, Lindenstrauss and Tzafriri [**17**, p. 72], [**16**, II.3.1].

Throughout this paper $\mathcal{E}$ and $\mathcal{F}$ denote families of finite dimensional Banach spaces and $\mathcal{E}^*$ denotes the family of the duals of the spaces in $\mathcal{E}$. We say that $\mathcal{E}$ is c-**chained to** $\mathcal{F}$, $c > 1$, if for each $E \in \mathcal{E}$, there is $G \in \mathcal{F}$ and bounded linear maps $T : E \to G$, $Q : G \to E$ with $\|T\| \le c, \|Q\| \le c$ such that $Q \circ T = id_E$.

Theorem 1. *Let $\mathcal{E}$, $\mathcal{F}$ be two families of finite dimensional Banach spaces such that $\mathcal{E}$ is c-chained to $\mathcal{F}$. If X is a Banach space λ-locally $\mathcal{E}-$factorable then X is λc-locally $\mathcal{F}-$factorable.*

Proof. Let A be a finite dimensional subspace of X, and consider a locally factorable diagram for A with respect to a companion $E_A \in \mathcal{E}$

$$\begin{array}{ccc} A & \overset{i_A}{\hookrightarrow} & X \\ \varphi_A \searrow & & \nearrow \eta_A \\ & E_A & \end{array}$$

By hypothesis, there is $G_A \in \mathcal{F}$ and bounded linear maps $E_A \xrightarrow{T} G_A \xrightarrow{Q} E_A$ such that $Q \circ T = id_{E_A}$ with $\|T\| \le c, \|Q\| \le c$. It is clear that the composition bounded linear maps $\psi_A = T \circ \phi_A : A \to G_A$ and $\beta_A = \eta_A \circ Q : G_A \to X$ satisfy $\beta_A \circ \psi_A = i_A$ with $\|\psi_A\| \le c\lambda, \|\beta_A\| \le c\lambda$. That is, the maps ψ_A, β_A are bounded linear factorization of i_A through $G_A \in \mathcal{F}$, proving that X is $c\lambda$-locally $\mathcal{F}-$factorable. □

Theorem 2. *Let X be a locally $\mathcal{E}-$factorable Banach space, and let Y be a locally complemented subspace of X. Then Y is locally $\mathcal{E}-$factorable.*

Proof. Let A be a finite dimensional subspace of Y, and consider a locally factorable diagram for A with respect to a companion $E_A \in \mathcal{E}$

$$\begin{array}{ccc} A & \overset{i_A}{\hookrightarrow} & X \\ \varphi_A \searrow & & \nearrow \eta_A \\ & E_A & \end{array}$$

Since Y is a locally complemented in X, and $\eta_A(E_A)$ is a finite dimensional subspace of X, there is a bounded operator $p_A : \eta_A(E_A) \to Y$ such that p_A is the identity on $\eta_A(E_A) \cap Y$, and hence is the identity on A, since $A = i_A(A) = \eta_A(\varphi_A(A)) \subseteq \eta_A(E_A) \cap Y$. Therefore, $i_A = p_A \circ i_A = \eta_A \circ \varphi_A$, proving that Y locally $\mathcal{E}-$factorable. □

Since any Banach space X is locally complemented in its bidual space X^{**}, we have

Corollary 1. *Any Banach space X has the same local factorable structure as X^{**}.*

The following Corollary is immediate since $\mathcal{L}_\infty$ spaces are locally complemented in any superspace.

Corollary 2. *All $\mathcal{L}_\infty$ spaces are locally $\mathcal{E}-$factorable whenever an $\mathcal{L}_\infty$ space is contained in a locally $\mathcal{E}-$factorable space.*

Corollary 3. *All $\mathcal{L}_1$ spaces are locally $\mathcal{E}^*-$factorable whenever an $\mathcal{L}_\infty$ space is contained in a locally $\mathcal{E}-$factorable space.*

Proof. It is easy to see that the family $\{\ell_\infty^n\}_{n=1}^\infty$ is chained to the family $\mathcal{E}$, and hence the family $\{\ell_1^n\}_{n=1}^\infty$is chained to the family $\mathcal{E}^*$. □

Example 1. *Recall that the Schreier Space S is the completion of the space of finite sequences with respect to the following norm:*

$$\|x\| = \sup_A \ \left(\Sigma_{j\in A}\,|x_j|\right),$$

where the supremum is taken over all "admissible" subsets of N, which are defined as the finite subsets $A = \{n_1, n_2, ..., n_k\}$ *of* $\mathbb{N}$ *such that* $n_1 < n_2 < ... < n_k$ *and* $k \leq n_1$, Schreier [**19**] (see also Castillo and González [**5**, p. 119])*. So, if S_k denotes the subspace of the Schreier space S generated by the first k elements of the canonical basis*$\{e_i\}_{i=1}^\infty$, *then* every $\mathcal{L}_\infty$ space (respectively, an $\mathcal{L}_1$ space) is locally $\{S_k\}_{k=1}^\infty-$factorable (respectively, locally $\{S_k^*\}_{k=1}^\infty$ *−factorable), since S contains isometic copies of the $\mathcal{L}_\infty$ space c_0* [**4**, p.167]**.**

Theorem 3. *Let X be a Banach space which is λ-locally $\mathcal{E}-$factorable and complemented in its bidual. Let $\mathcal{U}$ be an ultrafilter refining the order filter on the net $\mathcal{C}_X$ of the finite dimensional subspaces A of X. Then X is isomorphic to a complemented subspace of the ultraproduct $\left(\underset{A\in\mathcal{C}_X}{\Pi} E_A\right)_{\mathcal{U}}$ of all companions $E_A \in \mathcal{E}$ of $A \in \mathcal{C}_X$.*

Proof. For each $A \in \mathcal{C}_X$, consider a locally factorable diagram with respect to a companion $E_A \in \mathcal{E}$

$$\begin{tikzcd} A \arrow[rr, hook, "i_A"] \arrow[dr, "\varphi_A"'] & & X \\ & E_A \arrow[ur, "\eta_A"'] & \end{tikzcd}$$

Let $(y_A)_{\mathcal{U}} \in \left(\prod_{A\in\mathcal{C}_X} E_A\right)_{\mathcal{U}}$, then $\left(\|y_A\|_\infty\right)_{A\in\mathcal{C}_X}$ is a bounded net, since $(y_A)_{\mathcal{U}} = ((y_A) + N_{\mathcal{U}}) \in \ell_\infty\left(E_A;\mathcal{C}_X\right)/N_{\mathcal{U}}$, where

$$N_{\mathcal{U}} = \left\{(y_A)_{A\in\mathcal{C}_X} \in \ell_\infty\left(E_A;\mathcal{C}_X\right) : \lim_{\mathcal{U}} \|y_A\| = 0\right\}.$$

Therefore, $(\eta_A(y_A))_{A\in\mathcal{C}_X}$ is $\sigma(X^{**}, X^*)-$bounded in X^{**}, since it is bounded in $X \hookrightarrow X^{**}$. By Kadison and Ringrose [**13**, Corollary 1.6.6], the $\sigma(X^{**}, X^*)$-closure of $\{\eta_A(y_A) : A \in \mathcal{C}_X\}$ in X^{**} is compact, and hence, the weak-* limit of $(\eta_A(y_A))_{A\in\mathcal{C}_X}$ over $\mathcal{U}$ exists, by Sims [**20**, Lemma 3.2]. Accordingly, we can define a map $\Psi : \left(\prod_{A\in\mathcal{C}_X} E_A\right)_{\mathcal{U}} \longrightarrow X^{**}$ by

$$\Psi\left((y_A)_{\mathcal{U}}\right) = weak\ *\lim_{\mathcal{U}}(\eta_A(y_A)).$$

If is π is a projection of X^{**} onto X, and J_X is the natural isometric embedding of X into $\left(\prod_{A\in\mathcal{C}_X} A\right)_{\mathcal{U}}$, then the composition maps

$$X \overset{J_X}{\hookrightarrow} \left(\prod_{A\in\mathcal{C}_X} A\right)_{\mathcal{U}} \overset{\Phi}{\longrightarrow} \left(\prod_{A\in\mathcal{C}_X} E_A\right)_{\mathcal{U}} \overset{\Psi}{\longrightarrow} X^{**} \overset{\pi}{\longrightarrow} X$$

is clearly the identity map on X, where $\Phi : \left(\prod_{A\in\mathcal{C}_X} A\right)_{\mathcal{U}} \longrightarrow \left(\prod_{A\in\mathcal{C}_X} E_A\right)_{\mathcal{U}}$ is the map given by $\Phi\left((x_A)_{\mathcal{U}}\right) = (\varphi_A(x_A))_{\mathcal{U}}$. That is, $(\pi \circ \Psi)\circ(\Phi \circ J_X) = id_X$, proving the theorem. □

Using the proof of Theorem 3 with $\mathcal{E} = \left\{\ell_p^n\right\}_{n=1}^\infty$, the following collolary is obvious.

Corollary 4. *If a Banach space X is locally $\left\{\ell_p^n\right\}_{n=1}^\infty-$factorable, $1 \le p \le \infty$, then X is an $\mathcal{L}_p$ space (or an $\mathcal{L}_2$ if $1 < p < \infty$).*

Example 2. (*i*) *Consider* **the James space** *J, that is, the Banach space $(J, \|.\|)$ of all real sequences $x = (a_1, a_2, ...)$ such that $\lim_{n\to\infty} a_n = 0$ and $\sup\left(\sum_{i=1}^n \left(a_{p_{2i-1}} - a_{p_{2i}}\right)^2\right) < \infty$,where the supremum is taken over all choices of n and of positive integers $p_1 < p_2 < .... < p_{2n}$, equipped with the* *norm*

$$\|x\| = \sup \left(\sum_{i=1}^{n} \left(a_{p_{2i-1}} - a_{p_{2i}} \right)^2 \right)^{\frac{1}{2}}.$$

The unit vectors $\{e_n\}_{n=1}^{\infty}$ *form a basis of* J, *Fetter de Buen* [**9**, p. 12]. *For each* n, *let* $J_n = span\{e_1, ..., e_n\}$, *then* J *is* $(1+\epsilon)$*-locally* $\{J_n\}_{n=1}^{\infty}$ *by* [**11**, Lemma 1.4], *and hence, is locally* $\{J_n\}_{n=1}^{\infty}$ *−factorable .*

(*ii*) *Every separable Hilbert space is locally* $\{J_n\}_{n=1}^{\infty}$ *−factorable . Indeed,* ℓ_2 *is isomorphic to a complemented subspace of* J [**9**, Corollary 2.d.4], *and* J *is* $(1+\epsilon)$*-locally* $\{J_n\}_{n=1}^{\infty}$. *The result follows by* [**13**, 2.2.14]**.**

The generalization of [**2**, Theorem 2] to locally factorable spaces is given in the following:

Theorem 4. *Let* $\mathcal{E}$ *be a family of finite dimensional Banach spaces, and let* Y *be a Banach space complemented in its bidual. If* $Ext\,(W,Y) = 0$ *for some Banach space* W *containing* $\mathcal{E}$ *uniformly complemented, then* $Ext\,(X,Y) = 0$ *for any Banach space* X *which is locally* $\mathcal{E}$*−factorable.*

Proof. Let X be a Banach space which is λ-locally $\mathcal{E}-$factorable, and let $\mathcal{U}$ be an ultrafilter refining the ordered filter on the net $\mathcal{C}_X$ of the finite dimensional subspaces A *of* X. Let $F : X \to Y$ be a zero-linear map, and consider a a locally factorable diagram for $A \in \mathcal{C}_X$ with respect to $E_A \in \mathcal{E}$

$$\begin{array}{ccc} & A \overset{i_A}{\hookrightarrow} X & \\ \varphi_A & \searrow \; \nearrow & \eta_A \\ & E_A & \end{array}$$

Then $F \circ \eta_A : E_A \to Y$ is a z-linear map, and so there is, by [**11**, Lemma 1.6], a constant t, independent of A, and a linear map $L_A : E_A \to Y$ such that

$$\|F \circ \eta_A(y) - L_A(y)\| \leq tZ\,(F \circ \eta_A)\,\|y\|, \qquad y \in E_A.$$

Note that if $x \in X$, then $x = x_A \in A$ for some $A \in \mathcal{C}_X$, and so

$$\begin{aligned} \|L_A \circ \varphi_A(x_A)\| &\leq \|L_A \varphi_A(x_A) - F(\eta_A(\varphi_A(x_A)))\| + \|F(\eta_A(\varphi_A(x_A)))\| \\ &\leq tZ\,(F \circ \eta_A)\,\|\varphi_A(x_A)\| + \|F(x_A)\| \\ &\leq tZ\,(F)\,\|\varphi_A\|\,\|\eta_A\|\,\|x_A\| + \|F(x_A)\| \\ &\leq t\lambda^2 Z\,(F)\,\|x_A\| + \|F(x_A)\|. \end{aligned}$$

Let $\Phi : \left(\prod_{A\in\mathcal{C}_X} A\right)_{\mathcal{U}} \longrightarrow \left(\prod_{A\in\mathcal{C}_X} L_A(E_A)\right)_{\mathcal{U}}$ be the map given by $\Phi\left((x_A)_{\mathcal{U}}\right) = \left(L_A \circ \varphi_A(x_A)\right)_{\mathcal{U}}$. As in the proof of Theorem 3, the $\sigma(Y^{**}, Y^{*})-$limit of

$\left(L_A \circ \varphi_A (x_A)\right)_{A \in \mathcal{C}_X}$ over $\mathcal{U}$ exists in Y^{**}, and hence, we can define a map $\Psi : \left(\prod_{A \in \mathcal{C}_X} L_A(E_A)\right)_{\mathcal{U}} \longrightarrow Y^{**}$ by

$$\Psi\left(\left(L_A \circ \varphi_A (x_A)\right)_{\mathcal{U}}\right) = weak * \lim_{\mathcal{U}} (L_A \circ \varphi_A (x_A)).$$

Given a projection π of Y^{**} onto Y, and putting $L = \Psi \circ \Phi \circ J_X$, where J_X is the natural isometric embedding of X into $\left(\prod_{A \in \mathcal{C}_X} A\right)_{\mathcal{U}}$, we have for every $x \in X$

$$\begin{aligned}
\|F(x) - \pi L(x)\| &\leq \|\pi\| \, \|F(x) - L(x)\| \\
&= \|\pi\| \, \|F(x) - \Psi \circ \Phi((x_A)_{\mathcal{U}})\| \\
&= \|\pi\| \left\| weak * \lim_{\mathcal{U}} \left(F(x) - L_A \circ \varphi_A (x_A)\right) \right\| \\
&= \|\pi\| \left\| weak * \text{-}\lim_{\mathcal{U}} \left(F \circ \eta_A (\varphi_A (x_A)) - L_A (\varphi_A (x_A))\right) \right\| \\
&\leq t\lambda^2 Z(F) \, \|\pi\| \, \|x\|,
\end{aligned}$$

proving that F is trivial. □

Next we give a definition that includes colocality which can be inherited by complemented subspaces.

Definition 1. *Let $\mathcal{E}$ be a family of finite dimensional Banach spaces. A Banach space X is said to be* **λ-colocally $\mathcal{E}-$factorable** *(or simply* **colocally factorable** *$\mathcal{E}$) if there is a constant λ such that for every finite dimensional quotient B of X, there is $E_B \in \mathcal{E}$,* **a companion** of B, *such that the quotient map $q_B : X \to B$ factors to bounded linear maps $\psi_B : X \to E_B$ and $\gamma_B : E_B \to B$ with $\|\psi_B\| \leq \lambda$ and $\|\gamma_B\| \leq \lambda$ through E_B, the diagram*

$$\begin{array}{ccc} & X \xrightarrow{q_B} B & \\ \psi_B & \searrow \; \nearrow & \gamma_B \\ & E_B & \end{array}$$

is called ***a colocally factorable diagram*** *for B.*

Theorem 5. *A Banach space X is λ-colocally $\mathcal{E}$-factorable if and only if X^* is λ-locally $\mathcal{E}^*$-factorable.*

Proof. Suppose that X is λ-colocally $\mathcal{E}$-factorable and let B be a finite dimensional subspace of X^*. Since B is w^*-closed, $B = (X/A)^*$ for some closed subspace A of X which implies that B^* is a quotient of X, since

$B^* = (X/A)^{**} = X/A$. Consider a colocally factorable diagram for B^*, and then take adjoints of its maps

$$\begin{array}{ccc} X & \xrightarrow{q} & B^* \\ \psi \searrow & & \nearrow \gamma \\ & E_{B^*} & \end{array} \qquad\qquad \begin{array}{ccc} B = B^{**} & \xrightarrow{q^*} & X^* \\ \psi^* \searrow & & \nearrow \gamma^* \\ & E^*_{B^*} & \end{array}$$

we have a colocally factorable diagram for B, since $\psi^* \circ \gamma^* = q^* = i_B$, $\|\gamma^*\| = \|\gamma\| \leq \lambda$ and $\|\psi^*\| = \|\psi\| \leq \lambda$, proving that X^* is locally $\mathcal{E}^*$-factorable.

Conversely, suppose that X^* is λ-locally $\mathcal{E}^*$-factorable, and let $B = X/A$ be a finite dimensional quotient of X, then $B^* = (X/A)^* = A^\perp$ is a subspace of X^*, and the adjoint of the quotient map $q : X \longrightarrow B$ is the inclusion map $q^* : B^* \longrightarrow X^*$ of B^* into X^*, where $A^\perp$ is the annihilator of A (see Rudin [**18**, 4.7, 4.8]). Hence, there is $E^* \in \mathcal{E}^*$, and bounded linear maps $B^* \xrightarrow{\varphi} E^* \xrightarrow{\eta} X^*$ such that $\eta \circ \varphi = q^*$, with $\|\varphi\| \leq \lambda$ and $\|\eta\| \leq \lambda$. Taking adjoints of the maps we have

$$\begin{array}{ccc} X^{**} & \xrightarrow{q^{**}} & B^{**} \equiv B \\ \eta^* \searrow & & \nearrow \phi^* \\ & E^{**} \equiv E & \end{array}$$

It is easy to see that $\phi^* \circ \eta^* \mid_X = q^{**} \mid_X = q$, and so X is colocally $\mathcal{E}$-factorable.
□

The following Corollary is immediate by Corollary 2, and Theorem 5.

Corollary 5. (i) *All $\mathcal{L}_\infty$ spaces are colocally $\mathcal{E}$-factorable whenever an $\mathcal{L}_\infty$ space is contained in a locally $\mathcal{E}$-factorable space.*

(ii) *All $\mathcal{L}_1$ spaces are colocally* $\mathcal{E}^*$-factorable whenever *an $\mathcal{L}_\infty$ space is contained in a locally* $\mathcal{E}$-factorable space.

Theorem 6. *If the dual X^* of a Banach space X is λ-colocally $\mathcal{E}^*$-factorable, then X is $\lambda(1+\epsilon)$-locally $\mathcal{E}$-factorable, for every $\epsilon > 0$.*

Proof. Suppose that X^* is λ-colocally $\mathcal{E}^*$-factorable, and let A be a finite dimensional subspace of X. Then $A^* = X/A^\perp$ is a quotient of X^*, and hence, there is $E^* \in \mathcal{E}^*$, and bounded linear maps $X^* \xrightarrow{\psi} E^* \xrightarrow{\gamma} A^*$ such that $\gamma \circ \psi = q$ with $\|\psi\| \leq \lambda$ and $\|\gamma\| \leq \lambda$, where $q : X^* \longrightarrow A^*$ is the quotient map. Taking adjoints of the maps we have

$$\begin{array}{ccc} A \equiv A^{**} & \xrightarrow{q^*} & X^{**} \\ & \gamma^* \searrow \nearrow \psi^* & \\ & E^{**} = E & \end{array}$$

Since q^* is the inclusion map of $A \equiv A^{**}$ into X^{**}, we have $A = q^*(A) = \psi^*(\gamma^*(A)) \subseteq \psi^*(E)$. By the principle of local reflexivity, for every $\epsilon > 0$, there is an $(1+\epsilon)$-isomorphism $T : \psi^*(E) \longrightarrow X$ such that $Tx = x$ for all $x \in \psi^*(E) \cap X$, Johnson and Rosenthal **[12]**, which implies that $(T \circ \psi^*) \circ \gamma^* = q^* = i_A$. That is, the diagram

$$\begin{array}{ccc} A & \overset{i_A}{\hookrightarrow} & X \\ & \gamma^* \searrow \nearrow T \circ \psi^* & \\ & E & \end{array}$$

is a locally factorable diagram for A, proving that X is $\lambda(1+\epsilon)$-locally $\mathcal{E}$-factorable, since $\|T \circ \psi^*\| \leq \lambda(1+\epsilon)$. □

An immediate application of Theorem 5, and Theorem 6 we have:

Corollary 6. *Let X be a Banach space. If X^{**} is locally $\mathcal{E}$-factorable (resp. colocally $\mathcal{E}-$factorable), then X is locally $\mathcal{E}$-factorable (resp. colocally $\mathcal{E}$-factorable)*

Theorem 7. *Let X and Y be Banach spaces, and let $\psi_1 : X \longrightarrow Y, \psi_2 : Y \longrightarrow X$ be bounded linear operators such that $\psi_1 \circ \psi_2 = id_Y$. If X is colocally $\mathcal{E}$-factorable , so is Y.*

Proof. Suppose that X is colocally $\mathcal{E}$-factorable, then X^* is locally $\mathcal{E}^*$-factorable, by Theorem 5. Let B be a finite dimensional subspace of Y^*, then $A = \psi_1^*(B)$ is a finite dimensional subspace of X^*, hence there is $E^* \in \mathcal{E}^*$, and bounded linear maps $A \xrightarrow{\phi} E^* \xrightarrow{\eta} X^*$ such that $\eta \circ \varphi = i_A$. Put $\varphi_B = \varphi \circ \psi_1^*$ and $\eta_B = \psi_2^* \circ \eta$, where $\psi_1^* : Y^* \to X^*$ and $\psi_2^* : X^* \longrightarrow Y^*$ are the adjoint maps of ψ_1 and ψ_2, respectively. It is clear that the coposition map $\eta_B \circ \varphi_B$ is the identity operator id_{Y^*} on Y^*, and hence Y^* is locally $\mathcal{E}^*$-factorable, which implies that Y is colocally $\mathcal{E}$-factorable, by Theorem 5. □

Corollary 7. *If X is colocally $\mathcal{E}$-factorable, so is every complemented subspace of X, and every Banach space isomorphic to X.*

Example 3. *Every separable Hilbert space is colocally $\{\ell_p^n\}_{n=1}^{\infty}$-factorable,* $1 < p < \infty$, by **[13**, 2.2.13, 2.2.14] *and Corollary 7.*

The next theorem establishes a condition on the family $\mathcal{E}$ of finite dimensional Banach spaces, so that locally $\mathcal{E}$-factorable coincides with colocally $\mathcal{E}$-factorable.

Theorem 8. *Let X be a Banach space, and let $\mathcal{U}$ be an ultrafilter refining the order filter on the net $\mathcal{C}_X$ of all finite dimensional (resp. $\mathcal{F}_X$ of all closed finite codimensional) subspaces A of X. Suppose that:*

(i) the bidual $\left(\prod_{A\in\mathcal{C}_X} E_A\right)_{\mathcal{U}}^{**}$ of the ultraproduct $\left(\prod_{A\in\mathcal{C}_X} E_A\right)_{\mathcal{U}}$ of all companions $E_A \in \mathcal{E}$ of $A \in \mathcal{C}_X$ is colocally $\mathcal{E}$-factorable and,

(ii) the bidual $\left(\prod_{A\in\mathcal{F}_X} E_A^*\right)_{\mathcal{U}}^{**}$ of the ultraproduct $\left(\prod_{A\in\mathcal{F}_X} E_A^*\right)_{\mathcal{U}}$ is colocally $\mathcal{E}^*-$factorable, where $E_A \in \mathcal{E}$ is a companion of the quotient space X/A, $A \in \mathcal{F}_X$.

Then X is locally $\mathcal{E}$-factorable if and only if it is colocally $\mathcal{E}$-factorable.

Proof. Suppose that X is locally $\mathcal{E}$-factorable. Start as in the proof of Theorem 3, and consider the second adjoint operators

$$X^{**} \overset{(J_X)^{**}}{\hookrightarrow} \left(\left(\prod_{A\in\mathcal{C}_X} A\right)_{\mathcal{U}}\right)^{**} \xrightarrow{\Phi^{**}} \left(\left(\prod_{A\in\mathcal{C}_X} E_A\right)_{\mathcal{U}}\right)^{**} \xrightarrow{\Psi^{**}} X^{(4)}$$

of the maps

$$X \overset{J_X}{\hookrightarrow} \left(\prod_{A\in\mathcal{C}_X} A\right)_{\mathcal{U}} \xrightarrow{\Phi} \left(\prod_{A\in\mathcal{C}_X} E_A\right)_{\mathcal{U}} \xrightarrow{\Psi} X^{**}$$

Let $\pi : X^{(4)} \longrightarrow X^{**}$ be a projection, then it is easy to see that

$$(\pi \circ \Psi^{**})\circ(\Phi^{**}\circ(J_X)^{**}) = \pi\circ(\Psi \circ \Phi \circ J_X)^{**} = \pi\circ(i_X)^{**} = \pi\left(i_{X^{**}}\right) = id_{X^{**}},$$

where $X^{**} \overset{i_{X^{**}}}{\hookrightarrow} X^{(4)}$ is the inclusion map. Hence, X^{**} is isomorphic to a complemented subspace of $\left(\left(\prod_{A\in\mathcal{C}_X} E_A\right)_{\mathcal{U}}\right)^{**}$,which implies that X^{**} is colocally $\mathcal{E}$-factorable, by Corollary 7. Hence $(X^*)^{**} \cong (X^{**})^*$ is locally $\mathcal{E}^*-$factorable, by Theorem 5, and so X^* is locally $\mathcal{E}^*-$factorable, by Corollary 6, which implies that X is colocally $\mathcal{E}$-factorable, by Theorem 5.

For the converse, note first that $\left(\prod_{A\in\mathcal{F}_X} E_A^*\right)_{\mathcal{U}}$ is the ultraproduct of all companions in $\mathcal{E}^*$of finite dimensional subspaces B of X^*, since $B^* = (X/Z)^{**} = X/Z$ for some closed subspace Z of X. Thus, if X is colocally $\mathcal{E}$-factorable, then X^* is locally $\mathcal{E}^*$-factorable, by Theorem 5. Since $\left(\prod_{A\in\mathcal{F}_X} E_A^*\right)_{\mathcal{U}}^{**}$ is colocally $\mathcal{E}^*$-factorable, then X^* is colocally $\mathcal{E}^*$-factorable, by Theorem 8, and hence, X is locally $\mathcal{E}$-factorable, by Theorem 6. ☐

Example 4. A Banach space X is locally $\left\{\ell_p^n\right\}_{n=1}^{\infty}$-factorable if and only if it is colocally $\left\{\ell_p^n\right\}_{n=1}^{\infty}$-factorable, $1 \leq p \leq \infty$, *since the ultrapower of ℓ_p is reflexive.*

Theorem 9. *Let Y be a Banach space, and let $\mathcal{E}$ be a family of finite dimensional Banach spaces. If $Ext\,(Y,W)=0$ for some Banach space W containing $\mathcal{E}$ uniformly complemented, then $Ext\,(Y,Z)=0$ for every Banach space Z complemented in its bidual and colocally $\mathcal{E}$-factorable.*

Proof. Let Z be a Banach space complemented in its bidual which is λ-colocally factorable $\mathcal{E}$, and let $\mathcal{C}$ be the net of all finite codimensional subspaces A of Z directed by reverse inclusion. Let $\mathcal{U}$ be an ultrafilter which refines the corresponding order filter on $\mathcal{C}$, and note that $\left\{A^{\perp}:A\in\mathcal{C}\right\}$ is the net of all finite dimensional subspaces of Z^{*}, and $A^{\perp}\subseteq B^{\perp}$ when $B\subseteq A$, $A,B\in\mathcal{C}$. Let $F:Y\rightarrow Z$ be a z-linear map, and consider a pseudo-colocality diagram for Z/A, $A\in\mathcal{C}$ with respect to $E_{A}\in\mathcal{E}$

$$\begin{array}{ccc} Z & \xrightarrow{q_A} & Z/A \\ & \psi_A \searrow \quad \nearrow \gamma_A & \\ & E_A & \end{array}$$

It is clear that $\psi_A\circ F:Y\longrightarrow E_A$ is a z-linear map, and so there is, by [**11**, Lemma 1.6], a constant c, independent of A, and a linear map $L_A:Y\longrightarrow E_A$ such that

$$\left\|\psi_A\circ F\left(y\right)-L_A\left(y\right)\right\|\leq cZ\left(\psi_A\circ F\right)\left\|y\right\|\leq c\lambda Z\left(F\right)\left\|y\right\|,\qquad y\in E_A.$$

By Diestel [**7**, 8.8] and [**18**, 4.7, 4.8], there is a canonical isometric embedding $J:Z^{*}\rightarrow\left(\Pi_{A\in\mathcal{C}}A^{\perp}\right)_{\mathcal{U}}\equiv\left(\Pi_{A\in\mathcal{C}}\left(Z/A\right)^{*}\right)_{\mathcal{U}}$ given by $J\left(f\right)=\left(f_A\right)_{\mathcal{U}}$, $f\in Z^{*}$, where $f_A=f$ if $f\in A^{\perp}$ and $f_A=0$ otherwise. Therefore, by setting

$$\left(f_A\right)_{\mathcal{U}}\left(\left(z+A\right)_{\mathcal{U}}\right)=\lim_{\mathcal{U}}\left(f_A\left(z+A\right)\right),$$

$\left(\Pi_{A\in\mathcal{C}}\left(Z/A\right)^{*}\right)_{\mathcal{U}}$ embeds isometrically into $\left(\Pi_{A\in\mathcal{C}}\left(Z/A\right)\right)^{*}_{\mathcal{U}}$ (see [**7**, 8.3]), where the norm satisfies $\left\|\left(f_A\right)_{\mathcal{U}}\right\|=\lim_{\mathcal{U}}\left\|f_A\right\|$. If $Q:\left(\Pi_{A\in\mathcal{C}}\left(Z/A\right)\right)_{\mathcal{U}}\rightarrow Z^{**}$ is the restriction of the adjoint operator $J^{*}:\left(\Pi_{A\in\mathcal{C}}\left(Z/A\right)\right)^{**}_{\mathcal{U}}\rightarrow Z^{**}$ then

$$\left(Q\left(\left(z+A\right)_{\mathcal{U}}\right)\right)\left(f\right)=\left(J(f)\right)\left(\left(z+A\right)_{\mathcal{U}}\right)=\lim_{\mathcal{U}}f_A\left(z+A\right)=\lim_{\mathcal{U}}\left(f_A\left(z\right)\right),$$

for every $f\in Z^{*}$ and $\left(z+A\right)_{\mathcal{U}}\in\left(\Pi_{A\in\mathcal{C}_Z}\left(Z/A\right)\right)_{\mathcal{U}}$.

We claim that $Q\left(\left(q_AF\left(y\right)\right)_{\mathcal{U}}\right)=F\left(y\right)$. To see this, let $f\in Z^{*}$ and let A_f be an element in $\mathcal{C}_Z$ such that $f\in A_f^{\perp}$, then $f\in A^{\perp}$for every $A\in\mathcal{C}_Z$, $A\geq A_f$, so $f_A\left(F\left(y\right)\right)=f\left(F\left(y\right)\right).$ Therefore, considering any neighborhood V of $f\left(F\left(y\right)\right),$ then $\left\{A\in\mathcal{C}_Z:f_A\left(F\left(y\right)\right)\in V\right\}$ belongs to $\mathcal{U}$ since it contains $\left\{A\in\mathcal{C}_Z:A\geq A_f\right\}$, so that $\lim_{\mathcal{U}}\left(f_A\left(F\left(y\right)\right)\right)=f\left(F\left(y\right)\right)=F\left(y\right)\left(f\right)$, that is $Q_Z\left(\left(q_AF\left(y\right)\right)_{\mathcal{U}}\right)\left(f\right)=f(F\left(y\right))$. Let $L:Y\longrightarrow Z^{**}$ be the linear map defined by

$$L\left(y\right)=Q_Z\left(\gamma_AL_A\left(y\right)\right)_{\mathcal{U}}.$$

Since

$$\begin{aligned}\|\gamma_A L_A(y)\| &\le \|\gamma_A L_A(y) - \gamma_A\ \psi_A F(y)\| + \|\gamma_A\ \psi_A F(y)\| \\ &\le cm^2 Z(F)\|y\| + \|F(y)\|,\end{aligned}$$

then $(\gamma_A L_A(y))_A \in \ell_\infty(Z/A, \Omega)$. Moreover

$$\begin{aligned}\|\pi L(y) - F(y)\| &\le \|\pi\|\,\|Q_Z(\gamma_A L_A(y) - \gamma_A\ \psi_A F(y))_{\mathcal{U}}\| \\ &\le \|\pi\|\,\|Q_Z\|\, m \lim_{\mathcal{U}} \|L_A(y) - \psi_A F(y)\| \\ &\le \|\pi\|\,\|Q_Z\|\, m^2 c Z(F)\|y\|,\end{aligned}$$

where $\pi : Z^{**} \longrightarrow Z$ is a projection. So $dist(\pi L, F) < \infty$, proving that F is trivial. $\square$

Recall that two families $\mathcal{E}$ and $\mathcal{F}$ of finite dimensional Banach spaces are said to satisfy $Ext(\mathcal{F}, \mathcal{E}) = 0$ **uniformly** if there is a constant c such that, for every couple of spaces $A \in \mathcal{E}$ and $B \in \mathcal{F}$ and every 0-linear map $F : B \to A$, there is a linear map $L : B \to A$ such that $dist(F, L) \le cZ(F)$ **[2]**.

Theorem 10. *Let $\mathcal{E}$ and $\mathcal{F}$ be two families of finite dimensional Banach spaces such that $Ext(\mathcal{E}, \mathcal{F}) = 0$ uniformly. If Y and Z are Banach spaces such that Y is locally $\mathcal{E}$-factorable, Z is colocally $\mathcal{F}$-factorable and complemented in its bidual, then $Ext(Y, Z) = 0$.*

Proof. Suppose that Y is λ-locally $\mathcal{E}$-factorable and Z is c-colocally $\mathcal{F}$-factorable. Let G be a finite dimensional subspace of Y, and let A be a closed subspace of Z such that $\dim Z/A < \infty$. Then there are $E_G \in \mathcal{E}$ and $B_A \in \mathcal{F}$ such that the following locally factorable diagram for G and colocally factorable diagram for Z/A commute

$$\begin{array}{ccc} G & \overset{i_G}{\hookrightarrow} & Y \\ \varphi_G & \searrow\ \nearrow & \eta_G \\ & E_G & \end{array} \qquad\qquad \begin{array}{ccc} Z & \overset{q_A}{\longrightarrow} & Z/A \\ \psi_A & \searrow\ \nearrow & \gamma_A \\ & B_A & \end{array}$$

Given a z-linear map $F : Y \longrightarrow Z$, the composition map $E_G \overset{\eta_G}{\to} Y \overset{F}{\to} Z \overset{\psi_A}{\to} B_A$ is also a z-linear map. Since $Ext(\mathcal{E}, \mathcal{F}) = 0$ uniformly, there is a constant t and a linear map $L_{G,A} : E_G \longrightarrow B_A$ such that

$$\begin{aligned}\|\psi_A F \eta_G(x) - L_{G,A}(x)\| &\le tZ(\psi_A F \eta_G)\|x\| \\ &\le t\lambda c Z(F)\|x\|,\end{aligned}$$

for all $x \in E_G$. In particular, for all $y \in G$,

$$\left\|\psi_A F(y) - L_{G,A}\left(\varphi_G(y)\right)\right\| \le t\lambda^2 c Z(F)\|y\|.$$

Now let $\mathcal{V}$ be an ultrafilter refining the order filter on the net of all the finite dimensional subspaces of Y, and define $L_A : Y \longrightarrow (Z/A)^{**} = Z/A$ by

$$L_A(y) = w^*\text{-}\lim_{\mathcal{V}} (\gamma_A L_{G,A}(\varphi_G(y_G))), \qquad \text{where } y_G = \begin{cases} y & \text{if } y \in G \\ 0 & \text{otherwise} \end{cases}$$

Since

$$\begin{aligned} \|\gamma_A L_{G,A}(\varphi_G(y_G)) - \gamma_A \psi_A F(y_G)\| &\le \|\gamma_A\| \|\psi_A F(y_G) - L_{G,A}(\varphi_G(y_G))\| \\ &\le t\lambda^2 c^2 Z(F) \|y\|, \end{aligned}$$

then

$$\|\gamma_A L_{G,A}(\varphi_G(y_G))\| \le t\lambda^2 c^2 Z(F)\|y\| + \|F(y)\|,$$

so that the map L_A is well defined.

Next consider $\mathcal{C}_Z$,$\mathcal{U}$ and Q_Z as described in the proof of the Theorem 9, and define $L : Y \longrightarrow Z^{**}$ by $L(y) = Q_Z((L_A(y))_{\mathcal{U}})$. Hence,

$$\begin{aligned} \|\pi L(y) - F(y)\| &\le \|\pi\| \left\| Q_Z \left(w^*\text{-}\lim_{\mathcal{V}} \gamma_A L_{G,A}(\varphi_G(y_G)) - \gamma_A \psi_A F(y) \right)_{\mathcal{U}} \right\| \\ &\le \|\pi\| \|Q_Z\| t\lambda^2 c^2 Z(F) \|y\|, \end{aligned}$$

where $\pi : Z^{**} \longrightarrow Z$ is a projection. Hence F is trivial. □

A Banach spaces Y is said to be **finitely represented** in a Banach spaces X if for every $\lambda > 1$ and every finite dimensional subspace B of Y there is a subspace A of X which is λ-isomorphic to B. Typical examples of spaces finitely represented in X are X^{**} and ultrapowers X^U of X [**8**, Theorems 8.13, 8.16]. A Banach space Y is said to be **cofinitely represented** in a Banach space X if for each $\lambda > 1$ and every finite dimensional quotient B of Y, there is a finite dimensional quotient A of X which is λ-isomorphic to B, Díaz and Basallote [**6**]. It is known that a Banach space Y is finitely (resp. cofinitely) represented in a Banach space X if and only if $Y^*$is cofinitely (resp. finitely) represented in X^* [**6**, 3.5]. It is easy to see that if a Banach space Y is finitely represented (resp. cofinitely represented) in a Banach space X, then Y is $(1+\epsilon)$-locally (resp. -colocally) the family of all finite dimensional subspaces (resp. quotients) of X, for every $\epsilon > 0$.

Theorem 11. *Let X be a λ-locally $\mathcal{E}$-factorable Banach space. If for every finite dimensional subspace A of X, there is a companion $E_A \in \mathcal{E}$ of A such that $\eta_A(E_A) = A$, then any Banach space which is finitely represented in X is α-locally $\mathcal{E}\mathcal{E}$-factorable, for every $\alpha > \lambda$. In particular, X^* is α-colocally $\mathcal{E}^*$-factorable.*

Proof. Let Y be a Banach space finitely represented in X and let B be a finite dimensional subspace of Y. If $\alpha > \lambda$ is given, let $t = \frac{c}{\lambda}$ and let

$\psi_B : B \longrightarrow A$ be a t-isomorphism onto a subspace of X. Given a locally factorable diagram for A such that $\eta_{_A}(E_\alpha) = A$ we have the following commutative diagram:

$$\begin{array}{ccc} B & \overset{i_B}{\hookrightarrow} & Y \\ \psi_B \downarrow & & \uparrow \psi_B^{-1} \\ A & \longrightarrow & A \\ & \phi_A \searrow \nearrow \eta_A & \\ & E_A & \end{array}$$

which implies that i_B factors through E_A with $\|\phi_A \psi_B\| \leq \alpha$, $\left\|\psi_B^{-1}\eta_A\right\| \leq \alpha$.

The last sentence is immediate since X^{**} is finitely represented in X , and hence, it is locally $\mathcal{E}$-factorable, which implies that X^* is colocally $\mathcal{E}^*$-factorable, by Theorem 4. $\square$

Remark 1. Let X be a Banach space which is locally $\mathcal{E}-$factorable such that every finite dimensional subspace of X is c-complemented for some constant c. If A is a finite dimensional subspace of X, and

$$\begin{array}{ccc} A & \overset{i_A}{\hookrightarrow} & X \\ & \phi_A \searrow \nearrow \eta_A & \\ & E_A & \end{array}$$

is a locally factorable diagram for A, then $p_A \circ \eta_{_A}(E_A) = A$, where $p_A : X \to A$ is a projection of X onto A with $\|p_A\| \leq c$. Hence every Banach space Y finitely represented in X is locally $\mathcal{E}$-factorable.

Theorem 12. *Let X be a λ-colocally $\mathcal{E}$-factorable Banach space. If there is a constant m such that for every finite dimensional quotient A of X, there is a bounded linear operator $r_A : A \longrightarrow X$ such that $\|r_A\| \leq m$ and $q_A r_A = id_A$, then every Banach space Y cofinitely represented in X is c-colocally $\mathcal{E}$-factorable, for every $c > m\lambda$.*

Proof. Let Y be a Banach space cofinitely represented in X, and let B be a finite dimensional quotient of Y. Fix $c > m\lambda$, and let $t = \frac{c}{m\lambda}$, then there is a t-isomorphism η_B of B onto a quotient A of X. Consider a colocally factorable diagram for A

$$\begin{array}{ccc} X & \longrightarrow & A \\ & \psi_{_A} \searrow \nearrow \gamma_A & \\ & E_A & \end{array}$$

Then the diagram

$$\begin{array}{ccc} Y & \xrightarrow{q_B} & B \\ \psi_A r_A \eta_B q_B \searrow & & \nearrow \eta_B^{-1}\gamma_A \\ & E_A & \end{array}$$

is a colocally factorable diagram for B since $\left\|\psi_A r_A \eta_B q_B\right\| \leq c$, $\left\|\eta_B^{-1}\gamma_A\right\| \leq c$ and $\gamma_B \psi_B = q_B$, proving that Y is c-colocally $\mathcal{E}$-factorable. $\square$

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