Lyapunov Stability

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Introduction

- Stability is at the heart of any dynamical system
- There exist various kinds of stability in theory: Input-output stability, stability of periodic orbits, **stability of equilibrium points**, etc.
- Even in principle, different people have different perceived notion of stability
- What is stability? Why do you want it?

Introduction

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- What is stability? Why do you want it?

My notion of stability is the "*forgetfulness*": You want the system to forget where it has started from...

- You can design the system without worrying about the impact of initial condition or disturbance (usually unknown or unpredictable)
- Stability of a deterministic system
- Stability of a random system (e.g., Markov chain)

Stability of Equilibrium Points

Consider

$$\dot{x} = f(x) \tag{1}$$

- $x^* \in \mathbb{R}^n$ is an equilibrium point of (1), i.e., $f(x^*) = 0$
- Without loss of generality, we assume $x^* = 0$. If not, take $y = x - x^*$ and $g(y) = f(y + x^*)$ with $g(0) = 0$.
- Stability : little noise will die out or will not grow as time goes $\rightarrow$ you can 'steer' the system as wanted
- Will talk about stability of the origin $x = 0$.

Stability of Equilibrium Points

Definition: The equilibrium point $x = 0$ is

1. *stable* if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0$$

2. *unstable* if it is not stable
3. *asymptotically stable* if there exists a $\delta > 0$ such that

$$\lim_{t\to\infty} \|x(t)\| = 0, \quad \text{for all } \|x(0)\| < \delta.$$

4. *globally, asymptotically stable* if $\lim_{t\to\infty} \|x(t)\| = 0$ for all initial conditions $\|x(0)\|$.

Example of Stability of Equilibrium Points

Pendulum Example: An equation for pendulum dynamics can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - b x_2, \quad a > 0, b \geq 0 \end{aligned}$$

- Two equilibrium points: $(x_1^*, x_2^*) = (0, 0)$ and $(\pi, 0)$
- When $b = 0$ (neglecting friction), $(0, 0)$ is stable (trajectories around origin are closed orbits), but not asymptotically stable.
- When $b > 0$ (with friction), $(0, 0)$ is asymptotically stable (the pendulum eventually stops, or "energy" dissipates in the long run)
- $(\pi, 0)$ is saddle point (unstable)

Lyapunov Stability

Theorem: Consider a continuously differentiable function $V(x)$ such that $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$ (V is positive definite). We then have the following conditions for the various notions of stability.

1. If $\dot{V}(x) \leq 0$ for all x, then $x = 0$ is stable.
2. In addition, if $\dot{V}(x) < 0$ for all $x \neq 0$, then $x = 0$ is asymptotically stable.
3. In addition to (1) and (2) above, if V is *radially unbounded*, i.e.,

   $$\|x\| \to \infty \Rightarrow V(x) \to \infty,$$

   then $x = 0$ is globally asymptotically stable.

Lyapunov Stability

- V is called a *Lyapunov function*
- V must be positive definite.
- Stability means that the Lyapunov function decreases along the trajectory of $x(t)$.
- Case (1) means that $\dot{V}$ is negative semi-definite
- Case (2) means that $\dot{V}$ is negative definite
- This is only sufficient condition! Constructing Lyapunov functions is basically by trial-and-error.

Proof of Lyapunov Stability Theorem

Example of Lyapunov Stability

Consider the pendulum example:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - bx_2, \quad a > 0, b > 0, |x_1| < \pi \end{aligned}$$

Try $V(x) = a(1 - \cos x_1) + (1/2)x_2^2$ as a Lyapunov function candidate:

- Is V positive definite?
- Is $\dot{V}$ negative semi-definite? negative definite?

How about $V(x) = a(1 - \cos x_1) + (1/2)x^T P x$ for some 2×2 positive definite matrix P?

- Can we choose some P such that $\dot{V}$ is negative definite over some domain?

Existence and Uniqueness

Question: Does the solution of

$$\dot{x} = f(t, x), \quad x(0) = x_0 \tag{2}$$

exist? If so, is it unique?

- *Fact:* If $f(t, x)$ is continuous in its arguments, then there is at least one solution
- Consider $\dot{x} = x^{1/3}$ with $x(0) = 0$. Solution(s)?
- Continuity itself is not enough for uniqueness.

Lipschitz Condition

Definition: The function $f(t, x)$ is said to be **Lipschitz** continuous if

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \quad 0 < L < \infty. \tag{3}$$

for some range of t over some domain of x, y

- L: Lipschitz constant
- We say *locally Lipschitz* or *globally Lipschitz* to indicate the domain over which the Lipschitz condition (3) holds.

Lipschitz Condition

Special case: $f(t, x) = f(x)$ (depends only on x)

- $f(x)$ is *locally Lipschitz* on a domain $D \subset \mathbb{R}^n$ if each point of D has a neighborhood D_0 such that f satisfies (3) for all points in D_0 with some Lipschitz constant L_0 (possibly dependent on D_0).
- $f(x)$ is *Lipschitz* on a set W if it satisfies (3) for all points in W with the same Lipschitz constant L.
- A locally Lipschitz function on a domain $\not\Rightarrow$ Lipschitz on D, since the Lipschitz condition may not hold uniformly, i.e., $\sup_{D_0} L_0$ may be infinite where the supremum is taken over all neighborhoods of points in D.
- In a compact (closed and bounded) subset of D, locally Lipschitz $\Rightarrow$ Lipschitz
- $f(x)$ is *globally Lipschitz* if it is Lipschitz on $\mathbb{R}^n$.

Existence and Uniqueness of Solution to DE

Theorem: Suppose that $f(t,x)$ is piecewise continuous in t and satisfies the Lipschitz condition for all $x, y \in \mathbb{R}^n$, $\forall t \in [t_0, t_1]$. Then, (2) has a unique solution over $[t_0, t_1]$.

Theorem: Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and all $x \in D \subset \mathbb{R}^n$. Let W be a compact set of D, $x_0 \in W$, and suppose that every solution of

$$\dot{x} = f(t,x), \quad x(0) = x_0$$

lies entirely in W. Then, there is a unique solution defined for all $t \geq 0$.