

King Saud University
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Department of Mathematics

M-106
INTEGRAL CALCULUS

CLASS NOTES
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ANTIDERIVATIVES

Definition (Antiderivative): A function G is called an antiderivative of the function f on the interval I if $G'(x) = f(x)$ for all $x \in I$.

Example : What is the antiderivative of the function $f(x) = 2x$?

Answer : The antiderivative is $G(x) = x^2 + c$, where c is a constant.

Note: If $G_1(x)$ and $G_2(x)$ are both antiderivatives of the function $f(x)$ then $G_1(x) - G_2(x) = \text{constant}$.

Definition (indefinite integral): If $G(x)$ is the antiderivative of $f(x)$ then $\int f(x) dx = G(x) + c$, $\int f(x) dx$ is called the indefinite integral of the function $f(x)$.

Basic Rules of integration :

1. $\int 1 dx = x + c$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, where $n \neq -1$, $n \in \mathbb{Q}$
3. $\int \cos x dx = \sin x + c$
4. $\int \sin x dx = -\cos x + c$
5. $\int \sec^2 x dx = \tan x + c$
6. $\int \csc^2 x dx = -\cot x + c$
7. $\int \sec x \tan x dx = \sec x + c$
8. $\int \csc x \cot x dx = -\csc x + c$

Properties of indefinite integral :

1. $\int a f(x) dx = a \int f(x) dx$, where $a \in \mathbb{R}$
2. $\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$

Notes: If $G(x)$ is the antiderivative of the function $f(x)$ then

$$1. \int f(x) dx = G(x) + c$$

$$\int \frac{d}{dx} G(x) dx = G(x) + c$$

$$2. \frac{d}{dx} \int f(x) dx = f(x)$$

Example (1): Solve $\int \left(\frac{3}{x^4} - 5x \right) dx$

$$\begin{aligned} \text{Answer: } \int \left(\frac{3}{x^4} - 5x \right) dx &= \int (3x^{-4} - 5x) dx = \int 3x^{-4} dx - \int 5x dx \\ &= 3 \int x^{-4} dx - 5 \int x dx = 3 \frac{x^{-3}}{-3} - 5 \frac{x^2}{2} + c \end{aligned}$$

Example (2): Solve $\int \frac{2x^2 + 3}{\sqrt{x}} dx$

$$\begin{aligned} \text{Answer: } \int \frac{2x^2 + 3}{\sqrt{x}} dx &= \int \frac{2x^2 + 3}{x^{\frac{1}{2}}} dx \\ &= \int x^{-\frac{1}{2}} (2x^2 + 3) dx = \int \left(2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}} \right) dx \\ &= 2 \int x^{\frac{3}{2}} dx + 3 \int x^{-\frac{1}{2}} dx = 2 \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 3 \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \end{aligned}$$

CHANGE OF VARIABLE

Example: Solve $\int (4x + 1)^2 dx$

Put $u = 4x + 1$ then $du = 4 dx$, hence $\frac{1}{4} du = dx$

$$\int (4x + 1)^2 dx = \int u^2 \frac{1}{4} du = \frac{1}{4} \int u^2 du = \frac{1}{4} \frac{u^3}{3} + c = \frac{1}{4} \frac{(4x + 1)^3}{3} + c$$

Or we can use the form $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$, ($n \in \mathbb{Q}$, $n \neq -1$)

$$\int (4x + 1)^2 dx = \frac{1}{4} \int (4x + 1)^2 4 dx = \frac{1}{4} \frac{(4x + 1)^3}{3} + c$$

Where $f(x) = 4x + 1$, $n = 2$ and $f'(x) = 4$.

Basic Rules :

1. $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$, ($n \in \mathbb{Q}$, $n \neq -1$)
2. $\int \sin(f(x)) f'(x) dx = -\cos(f(x)) + c$
3. $\int \cos(f(x)) f'(x) dx = \sin(f(x)) + c$
4. $\int \sec^2(f(x)) f'(x) dx = \tan(f(x)) + c$
5. $\int \csc^2(f(x)) f'(x) dx = -\cot(f(x)) + c$
6. $\int \sec(f(x)) \tan(f(x)) f'(x) dx = \sec(f(x)) + c$
7. $\int \csc(f(x)) \cot(f(x)) f'(x) dx = -\csc(f(x)) + c$

Examples :

1. $\int \cos(3x + 4) dx = \frac{1}{3} \int \cos(3x + 4) 3 dx = \frac{1}{3} \sin(3x + 4) + c$
2. $\int \left(1 + \frac{5}{x}\right)^3 \frac{1}{x^2} dx = \frac{-1}{5} \int \left(1 + \frac{5}{x}\right)^3 \frac{-5}{x^2} dx = \frac{-1}{5} \frac{\left(1 + \frac{5}{x}\right)^4}{4} + c$
3. $\int \sqrt{9 - x^2} x dx = \frac{-1}{2} \int (9 - x^2)^{\frac{1}{2}} (-2x) dx = \frac{-1}{2} \frac{(9 - x^2)^{\frac{3}{2}}}{\frac{3}{2}} + c$
4. $\int \frac{1}{\sqrt{x} (1 + \sqrt{x})^3} dx = 2 \int (1 + \sqrt{x})^{-3} \frac{1}{2\sqrt{x}} dx = 2 \frac{(1 + \sqrt{x})^{-2}}{-2} + c$
5. $\int \tan^2 x \sec^2 x dx = \int (\tan x)^2 \sec^2 x dx = \frac{(\tan x)^3}{3} + c$

$$6. \int \frac{1}{\cos^3 x \csc x} dx = \int (\cos x)^{-3} \sin x dx = - \int (\cos x)^{-3} (-\sin x) dx$$

$$= -\frac{(\cos x)^{-2}}{-2} + c$$

$$7. \int \frac{\sin(1 + \sqrt{x})}{\sqrt{x}} dx = 2 \int \sin(1 + \sqrt{x}) \frac{1}{2\sqrt{x}} dx = -2 \cos(1 + \sqrt{x}) + c$$

$$8. \int \frac{\cos(\sqrt[3]{x})}{\sqrt[3]{x^2}} dx = 3 \int \cos(x^{\frac{1}{3}}) \frac{1}{3} x^{-\frac{2}{3}} dx = 3 \sin(x^{\frac{1}{3}}) + c$$

$$9. \int \frac{\cos \sqrt{x}}{\sqrt{x} \sin^2 \sqrt{x}} dx = 2 \int (\sin \sqrt{x})^{-2} \cos(\sqrt{x}) \frac{1}{2\sqrt{x}} dx$$

$$= 2 \frac{(\sin \sqrt{x})^{-1}}{-1} + c$$

Another Solution : $\int \frac{\cos \sqrt{x}}{\sqrt{x} \sin^2 \sqrt{x}} dx = \int \frac{1}{\sin \sqrt{x}} \frac{\cos \sqrt{x}}{\sin \sqrt{x}} \frac{1}{\sqrt{x}} dx$

$$= 2 \int \csc \sqrt{x} \cot \sqrt{x} \frac{1}{2\sqrt{x}} dx = -2 \csc \sqrt{x} + c$$

10. Find the value of k that satisfies $\int \sqrt{2x+3} dx = k(2x+3)^{\frac{3}{2}} + c$

$$\frac{d}{dx} \left[k(2x+3)^{\frac{3}{2}} + c \right] = \sqrt{2x+3}$$

$$\frac{3}{2}k(2x+3)^{\frac{1}{2}} \cdot 2 = (2x+3)^{\frac{1}{2}}$$

$$3k = 1, \text{ and hence } k = \frac{1}{3}$$

SUMS AND SIGMA NOTATION

If $a_1, a_2, \dots, a_n \in \mathbb{R}$ then $\sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n$.

Theorem : If $c, a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$ then

1. $\sum_{i=1}^n c = cn$.
2. $\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$.
3. $\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$.
4. $\sum_{i=1}^n i = \frac{n(n+1)}{2}$.
5. $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.
6. $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Examples :

1.
$$\begin{aligned} \sum_{k=1}^4 (k^3 - k + 2) &= \sum_{k=1}^4 k^3 - \sum_{k=1}^4 k + \sum_{k=1}^4 2 \\ &= \left(\frac{4(4+1)}{2}\right)^2 - \frac{4(4+1)}{2} + 2(4) = 100 - 10 + 8 = 98. \end{aligned}$$
2.
$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{5k}{n^2} = \lim_{n \rightarrow \infty} \frac{5}{n^2} \sum_{k=1}^n k = \lim_{n \rightarrow \infty} \frac{5}{n^2} \frac{n(n+1)}{2} = \frac{5}{2}.$$
3.
$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n^3} (i-1)^2 &= \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{i=1}^n (i^2 - 2i + 1) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^3} \left[\sum_{i=1}^n i^2 - 2 \sum_{i=1}^n i + \sum_{i=1}^n 1 \right] = \lim_{n \rightarrow \infty} \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6} - 2 \frac{n(n+1)}{2} + n \right] \\ \lim_{n \rightarrow \infty} \left[\frac{n(n+1)(2n+1)}{6n^3} - \frac{n(n+1)}{n^3} + \frac{n}{n^3} \right] &= \frac{2}{6} - 0 + 0 = \frac{1}{3} \end{aligned}$$

RIEMANN SUM

In this section we assume that the function $f(x) \geq 0$ on the interval $[a, b]$.

Definition (Regular Partition) : The set $\{x_0 = a, x_1, \dots, x_n = b\}$ is called a regular partition of the interval $[a, b]$ if $x_i = x_0 + i \Delta x$ for every $i = 1, 2, \dots, n$, and $\Delta x = \frac{b-a}{n}$.

This regular partition divides the interval $[a, b]$ into n subintervals of the form $[x_{i-1}, x_i]$ where $i = 1, 2, \dots, n$

Area under the graph of a function :

If $f(x) \geq 0$ on the interval $[a, b]$ and $\{x_0 = a, x_1, \dots, x_n = b\}$ is a regular partition of $[a, b]$, then the area under the graph of $f(x)$ can be approximated by n

rectangles using the formula $A_n = \sum_{i=1}^n f(x_i) \Delta x$

Example : Approximate the area under the graph of $f(x) = 2x - 2x^2$ on the interval $[0, 1]$ using 10 rectangles .

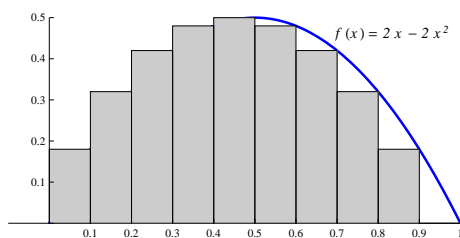
Answer : $\Delta x = \frac{1-0}{10} = 0.1$.

$x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, \dots , $x_9 = 0.9$, $x_{10} = 1$

$$A_{10} = \sum_{i=1}^{10} f(x_i) \Delta x = \sum_{i=1}^{10} (2x_i - 2x_i^2) 0.1$$

$$A_{10} = 0.1 [0.18 + 0.32 + 0.42 + 0.48 + 0.5 + 0.48 + 0.42 + 0.32 + 0.18 + 0]$$

$$A_{10} = 0.1(3.3) = 0.33$$



Definition (Riemann Sum) :

Let $\{x_0 = a, x_1, \dots, x_n = b\}$ be a regular partition of the interval $[a, b]$ with $\Delta x = \frac{b-a}{n}$. Pick points c_1, c_2, \dots, c_n where c_i is any point in the subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \dots, n$.

The Riemann sum is $R_n = \sum_{i=1}^n f(c_i) \Delta x$.

The area under the curve of $f(x)$ is the limit of the Riemann sum.

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x.$$

Example 1: Find the area under the curve of the function $f(x) = 3x + 1$ on the interval $[1, 3]$ using Riemann sum and c_i is the middle point of the subinterval.

Answer: $\Delta x = \frac{3-1}{n} = \frac{2}{n}$

$x_0 = 1, x_i = x_0 + i\Delta x = 1 + \frac{2i}{n}$ for every $i = 1, 2, \dots, n$.

For every $i = 1, 2, \dots, n, c_i \in [x_{i-1}, x_i], c_i = \frac{x_i + x_{i-1}}{2} = \frac{\left(1 + \frac{2i}{n}\right) + \left(1 + \frac{2(i-1)}{n}\right)}{2}$

$c_i = \frac{2 + (2i-1)\frac{2}{n}}{2} = 1 + \frac{2i-1}{n}$.

$$\begin{aligned} R_n &= \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n \left[3 \left(1 + \frac{2i-1}{n} \right) + 1 \right] \frac{2}{n} \\ &= \frac{2}{n} \sum_{i=1}^n \left[3 + \frac{6i-3}{n} + 1 \right] = \frac{2}{n} \sum_{i=1}^n \left[4 + \frac{6i}{n} - \frac{3}{n} \right] \\ &= \frac{2}{n} \left[\sum_{i=1}^n 4 + \frac{6}{n} \sum_{i=1}^n i - \frac{1}{n} \sum_{i=1}^n 3 \right] = \frac{2}{n} \left[4n + \frac{6}{n} \frac{n(n+1)}{2} - \frac{1}{n} 3n \right] \\ &= 8 + 6 \frac{n(n+1)}{n^2} - \frac{6}{n}. \end{aligned}$$

The desired area = $\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \left[8 + 6 \frac{n(n+1)}{n^2} - \frac{6}{n} \right] = 8 + 6 - 0 = 14$

Example 2: Do the last example where c_i is the end point of the subinterval.

Answer: For every $i = 1, 2, \dots, n, c_i \in [x_{i-1}, x_i], c_i = x_i = 1 + \frac{2i}{n}$

$$\begin{aligned} R_n &= \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n \left[3 \left(1 + \frac{2i}{n} \right) + 1 \right] \frac{2}{n} \\ &= \frac{2}{n} \sum_{i=1}^n \left[3 + \frac{6i}{n} + 1 \right] = \frac{2}{n} \sum_{i=1}^n \left[4 + \frac{6i}{n} \right] \\ &= \frac{2}{n} \left[\sum_{i=1}^n 4 + \frac{6}{n} \sum_{i=1}^n i \right] = \frac{2}{n} \left[4n + \frac{6}{n} \frac{n(n+1)}{2} \right] = 8 + 6 \frac{n(n+1)}{n^2} \end{aligned}$$

The desired area = $\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \left[8 + 6 \frac{n(n+1)}{n^2} \right] = 8 + 6 = 14$

THE DEFINITE INTEGRAL

Definition (The definite Integral) : For any continuous function f defined on the interval $[a, b]$ the definite integral of f from a to b is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x, \text{ whenever the limit exists.}$$

(where c_i is any point in the subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \dots, n$).

Notes :

1. Riemann Sum is the same for any choice of the points c_1, c_2, \dots, c_n .
2. When the limit exists we say that the function f is integrable.

Notes : If the function f is continuous on $[a, b]$ and $f(x) \geq 0$ for every $x \in [a, b]$, then

1. $\int_a^b f(x) dx \geq 0$.
2. $\int_a^b f(x) dx =$ The area under the curve of f

Example 1:

$$\int_1^3 (3x + 1) dx = \text{Area under the curve of } f = \lim_{n \rightarrow \infty} R_n = 14.$$

(See the example on Riemann sum).

Example 2: The definite integral representing $\lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{x_k + 1} \Delta x$ using regular

partition of the interval $[1, 2]$ is $\int_1^2 \sqrt{x+1} dx$.

Theorem: If the function f is continuous on the interval $[a, b]$ then f is integrable on $[a, b]$.

Properties of the definite integral : If the functions f and g are integrable on $[a, b]$ then :

1. $\int_a^b k f(x) dx = k \int_a^b f(x) dx$, for every $k \in \mathbb{R}$.
2. $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$.
3. For every $c \in [a, b]$ $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$.

4. If $f(x) \leq g(x)$ for every $x \in [a, b]$ then $\int_a^b f(x) dx \leq \int_a^b g(x) dx$

Example 3:

1. $\int_2^7 (x^2 - 3) dx - \int_2^4 (x^2 - 3) dx = \int_4^7 (x^2 - 3) dx$.

2. Since $\cos x \geq \sin x$ for every $x \in \left[0, \frac{\pi}{4}\right]$ then $\int_0^{\frac{\pi}{4}} \cos x dx \geq \int_0^{\frac{\pi}{4}} \sin x dx$.

3. To show that $\int_{-1}^1 \frac{x^2}{x^2 + 4} dx \leq \int_{-1}^1 x^2 dx$

For every $x \in [-1, 1]$, $x^2 + 4 > 1 \Rightarrow \frac{1}{x^2 + 4} < 1 \Rightarrow \frac{x^2}{x^2 + 4} \leq x^2$

Hence $\int_{-1}^1 \frac{x^2}{x^2 + 4} dx \leq \int_{-1}^1 x^2 dx$.

FUNDAMENTAL THEOREM OF CALCULUS

Fundamental Theorem of Calculus (Part I) :

If f is a continuous function on the interval $[a, b]$ and $G(x)$ is the antiderivative of $f(x)$ on $[a, b]$ then $\int_a^b f(x) dx = [G(x)]_a^b = G(b) - G(a)$.

Note : $\int_a^b \frac{d}{dx} G(x) dx = G(b) - G(a)$.

Examples :

1. $\int_0^2 (x^2 - 2x) dx = \left[\frac{x^3}{3} - x^2 \right]_0^2 = \left(\frac{8}{3} - 4 \right) - \left(\frac{0}{3} - 0 \right) = -\frac{4}{3}$.

2. Find the area under the graph of $f(x) = \sin x$ on $[0, \pi]$

Answer : The area = $\int_0^\pi \sin x dx = [-\cos x]_0^\pi = (-\cos \pi) - (-\cos 0) = 2$

Fundamental Theorem of Calculus (Part II) :

If f is a continuous function on the interval $[a, b]$ and $G(x) = \int_a^x f(t) dt$ for every $x \in [a, b]$ then $G'(x) = f(x)$ for every $x \in [a, b]$

Note : $G(x)$ is the antiderivative of $f(x)$ on $[a, b]$.

Examples :

1. $\frac{d}{dx} \int_0^x \sqrt{t^2 + 1} dt = \sqrt{x^2 + 1}$.

2. $\frac{d}{dx} \int_1^x \frac{1}{t^2 + 1} dt = \frac{1}{x^2 + 1}$.

3. $\frac{d}{dx} \int_3^x \left(2 + \frac{d}{dt} \cos t \right) dt = \frac{d}{dx} \int_3^x (2 - \sin t) dt = 2 - \sin x$

4. $\frac{d}{dt} \int_2^t \frac{1}{x^3 + 5} dx = \frac{1}{t^3 + 5}$

Theorem :

If f is a continuous function, g and h are differentiable functions then

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x)) h'(x) - f(g(x)) g'(x).$$

Notes :

1. If $g(x) = a$ and $h(x) = b$ then $\frac{d}{dx} \int_a^b f(t) dt = f(b)(0) - f(a)(0) = 0$
2. If $g(x) = a$ and $h(x) = x$ then $\frac{d}{dx} \int_a^x f(t) dt = f(x)(1) - f(a)(0) = f(x)$

Examples :

1. Find $G'(x)$, if $G(x) = \int_{1-x}^{x^2} \frac{1}{4+3t^2} dt$.
Answer : $G'(x) = \frac{d}{dx} \int_{1-x}^{x^2} \frac{1}{4+3t^2} dt = \frac{1}{4+3(x^2)^2}(2x) - \frac{1}{4+3(1-x)^2}(-1)$
 $G'(x) = \frac{2x}{4+3x^4} + \frac{1}{4+3(1-x)^2}$
2. $\frac{d}{dt} \left[\int_2^t \sqrt{x^2+1} dx + \int_t^{-1} \sqrt{x^2+1} dx \right] = \frac{d}{dt} \int_2^{-1} \sqrt{x^2+1} dx = 0$
3. Find $F'(2)$, if $F(x) = \int_1^{x^2} \frac{1}{t} dt$.
Answer : $F'(x) = \frac{d}{dx} \int_1^{x^2} \frac{1}{t} dt = \frac{1}{x^2}(2x) - 0 = \frac{2x}{x^2} = \frac{2}{x}$.
Hence $F'(2) = \frac{2}{2} = 1$.
4. Find $f(4)$, if $\int_0^x f(t) dt = x \cos \pi x$
Answer : Differentiate both sides with respect to x
 $\frac{d}{dx} \int_0^x f(t) dt = \frac{d}{dx} [x \cos \pi x]$
 $f(x) = (1) \cos \pi x + x (-\sin \pi x) \pi = \cos \pi x - \pi x \sin \pi x$
Hence $f(4) = \cos 4\pi - 4\pi \sin 4\pi = 1 - 4\pi(0) = 1$.
5. $\int_{-x}^x \frac{d}{dt} f(t) dt = f(x) - f(-x)$
Here, we used $\int_a^b \frac{d}{dx} G(x) dx = G(b) - G(a)$

Exercises: Solve the following :

1. $\frac{d}{dx} \int_0^5 \sqrt{t^2+3} dt$.
2. $\frac{d}{dx} \int_x^1 u^2 \cos u du$.
3. Find $F'(0)$, if $F(x) = \int_x^{x^2} \frac{1}{t-1} dt$.

AVERAGE VALUE OF A FUNCTION

Definition (Average value of a function) : Let f be a continuous function

on $[a, b]$ then the average value of f on $[a, b]$ is $f_{av} = \frac{\int_a^b f(x) dx}{b - a}$.

Example : Find f_{av} of the following functions :

1. $f(x) = x^2 - 2x$ on the interval $[1, 4]$

$$\begin{aligned} \int_1^4 (x^2 - 2x) dx &= \left[\frac{x^3}{3} - x^2 \right]_1^4 \\ &= \left(\frac{64}{3} - 16 \right) - \left(\frac{1}{3} - 1 \right) = \frac{63}{3} - 15 = \frac{63 - 45}{3} = \frac{18}{3} = 6 \end{aligned}$$

$$\text{Hence } f_{av} = \frac{\int_1^4 (x^2 - 2x) dx}{4 - 1} = \frac{6}{3} = 2 .$$

2. $f(x) = \sin^2 x \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^2 x \cos x dx &= \int_0^{\frac{\pi}{2}} (\sin x)^2 \cos x dx = \left[\frac{(\sin x)^3}{3} \right]_0^{\frac{\pi}{2}} \\ &= \frac{(\sin \frac{\pi}{2})^3}{3} - \frac{(\sin 0)^3}{3} = \frac{1}{3} - 0 = \frac{1}{3} \end{aligned}$$

$$\text{Hence } f_{av} = \frac{\int_0^{\frac{\pi}{2}} \sin^2 x \cos x dx}{\frac{\pi}{2} - 0} = \frac{\frac{1}{3}}{\frac{\pi}{2}} = \frac{2}{3\pi} .$$

Exercise : Find f_{av} of the function $f(x) = (2x + 1)^2$ on the interval $[0, 1]$

INTEGRAL MEAN VALUE THEOREM

Theorem (Integral Mean Value Theorem) :

If f is a continuous function on the interval $[a, b]$ then there exists a number

$$c \in (a, b) \text{ for which } f(c) = \frac{\int_a^b f(x) dx}{b-a} .$$

Example : Find the value that satisfies the integral Mean value theorem for the function $f(x) = 4x^3 - 1$ on the interval $[1, 2]$

$$\text{Answer : } f(c) = \frac{\int_1^2 (4x^3 - 1) dx}{2 - 1}$$

$$4c^3 - 1 = [x^4 - x]_1^2$$

$$4c^3 - 1 = (16 - 2) - (1 - 1)$$

$$4c^3 - 1 = 14$$

$$c^3 = \frac{15}{4}$$

$$c = \sqrt[3]{\frac{15}{4}}$$

Note that $c = \sqrt[3]{\frac{15}{4}} \in (1, 2)$.

NUMERICAL INTEGRATION

1. The Trapezoidal Rule :

It is used to approximate $\int_a^b f(x) dx$ with a regular partition of the interval $[a, b]$, where $\Delta x = \frac{b-a}{n}$, by using the formula

$$\int_a^b f(x) dx \approx \frac{b-a}{2n} [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Example : Approximate the integral $\int_0^1 \sqrt{x+x^2} dx$ using Trapezoidal rule with $n = 4$.

Answer : $[a, b] = [0, 1]$, $f(x) = \sqrt{x+x^2}$ and $\Delta x = \frac{1-0}{4} = 0.25$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	0	1	0
1	0.25	0.559017	2	1.11803
2	0.5	0.86625	2	1.73205
3	0.75	1.14564	2	2.29129
4	1	1.41421	1	1.41421
				6.55559

$$\int_0^1 \sqrt{x+x^2} dx \approx \frac{1-0}{2(4)} [f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)]$$

$$\int_0^1 \sqrt{x+x^2} dx \approx \frac{1}{8} [6.55559] \approx 0.819448 .$$

Exercise : Approximate the integral $\int_2^4 \frac{1}{x-1} dx$ using Trapezoidal rule with $n = 4$.

2. Simpson's Rule :

It is used to approximate $\int_a^b f(x) dx$ with a regular partition of the interval $[a, b]$, where $\Delta x = \frac{b-a}{n}$, and n is even, by using the formula

$$\int_a^b f(x) dx \approx \frac{b-a}{3n} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Example : Approximate the integral $\int_0^{10} \sqrt{10x - x^2} dx$ using Simpson's rule with $n = 4$.

Answer : $[a, b] = [0, 10]$, $f(x) = \sqrt{10x - x^2}$ and $\Delta x = \frac{10-0}{4} = 2.5$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	0	1	0
1	2.5	4.33013	4	17.3204
2	5	5	2	10
3	7.5	4.33013	4	17.3204
4	10	0	1	0
				44.6408

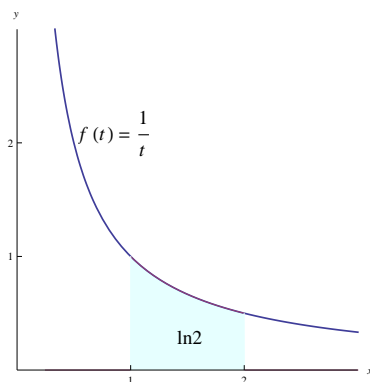
$$\int_0^{10} \sqrt{10x - x^2} dx \approx \frac{10-0}{3(4)} [f(0) + 4f(2.5) + 2f(5) + 4f(7.5) + f(10)]$$
$$\int_0^1 \sqrt{10x - x^2} dx \approx \frac{10}{12} [44.6408] \approx 37.2007 .$$

Exercise : Approximate the integral $\int_0^2 \frac{x}{x+1} dx$ using Simpson's rule with $n = 4$.

THE NATURAL LOGARITHMIC FUNCTION

Definition (The natural logarithmic function) :

For $x > 0$, the natural logarithmic function is defined by $\ln x = \int_1^x \frac{1}{t} dt$.



Note : The domain of the function $\ln x$ is the open interval $(0, \infty)$

Example : What is the domain of the function $\ln(x - 2)$?

Answer : $x - 2 > 0 \Rightarrow x > 2 \Rightarrow$ the domain is $(2, \infty)$.

Notes :

1. If $x > 1$ then $\ln x > 0$.
2. $\ln 1 = 0$.
3. If $0 < x < 1$ then $\ln x < 0$.

The graph of $\ln x$:

1. First derivative test :

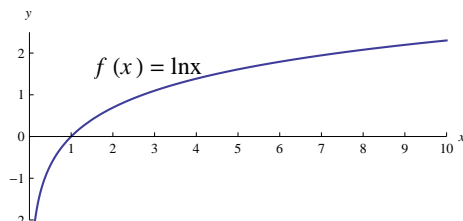
$$\frac{d}{dx} \ln x = \frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x} > 0 \text{ for every } x \in (0, \infty).$$

Hence $\ln x$ is an increasing function on $(0, \infty)$.

2. Second derivative test :

$$\frac{d^2}{dx^2} \ln x = \frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2} < 0 \text{ for every } x \in (0, \infty).$$

Hence $\ln x$ is a convex function on $(0, \infty)$.



Notes :

1. The range of the function $\ln x$ is \mathbb{R} .
2. $\lim_{x \rightarrow \infty} \ln x = \infty$.
3. $\lim_{x \rightarrow 0^+} \ln x = -\infty$.

The derivative of $\ln |x|$:

1. $\frac{d}{dx} \ln |x| = \frac{1}{x}$.
2. $\frac{d}{dx} \ln |f(x)| = \frac{f'(x)}{f(x)}$.

Note : $\ln |x|$ is the antiderivative of $\frac{1}{x}$.

Integration :

1. $\int \frac{1}{x} dx = \ln |x| + c$.
2. $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$.

Some properties of $\ln |x|$: If $x, y > 0$ and $r \in \mathbb{R}$ then

1. $\ln(xy) = \ln x + \ln y$.
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$.
3. $\ln x^r = r \ln x$.

Examples :

1. Simplify $\frac{1}{5} [2 \ln |x+1| + \ln |x| - \ln |x^2-2|]$
$$\frac{1}{5} [2 \ln |x+1| + \ln |x| - \ln |x^2-2|] = \frac{1}{5} [\ln(x+1)^2 + \ln |x| - \ln |x^2-2|]$$
$$= \frac{1}{5} [\ln |x(x+1)^2| - \ln |x^2-2|] = \frac{1}{5} \ln \left| \frac{x(x+1)^2}{x^2-2} \right| = \ln \left| \left(\frac{x(x+1)^2}{x^2-2} \right)^{\frac{1}{5}} \right|$$
2. If $y = \sqrt{\frac{(x+1)^4(x+2)^3}{(x-1)^2}}$ then find y' .
$$\ln y = \ln \left| \sqrt{\frac{(x+1)^4(x+2)^3}{(x-1)^2}} \right| = \frac{1}{2} [4 \ln |x+1| + 3 \ln |x+2| - 2 \ln |x-1|]$$

Differentiate both sides

$$\frac{y'}{y} = \frac{1}{2} \left[4 \frac{1}{x+1} + 3 \frac{1}{x+2} - 2 \frac{1}{x-1} \right]$$

$$\text{Hence } y' = \frac{1}{2} \sqrt{\frac{(x+1)^4(x+2)^3}{(x-1)^2}} \left[\frac{4}{x+1} + \frac{3}{x+2} - \frac{2}{x-1} \right]$$

Exercise : If $f(x) = \frac{x^2(2x-1)^3}{(x+5)^2}$ then find $f'(x)$?

More Basic Rules of Integration :

1. $\int \tan x \, dx = \ln |\sec x| + c .$
2. $\int \cot x \, dx = \ln |\sin x| + c .$
3. $\int \sec x \, dx = \ln |\sec x + \tan x| + c .$
4. $\int \csc x \, dx = \ln |\csc x - \cot x| + c$

Examples :

$$1. \int \frac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x} \, dx = \frac{1}{3} \int \frac{3x^2 + 6x + 9}{x^3 + 3x^2 + 9x} \, dx = \frac{1}{3} \ln |x^3 + 3x^2 + 9x| + c .$$

$$2. \int \frac{x^2 + 2x + 3}{(x^3 + 3x^2 + 9x)^5} \, dx = \frac{1}{3} \int (x^3 + 3x^2 + 9x)^{-5} (3x^2 + 6x + 9) \, dx \\ = \frac{1}{3} \frac{(x^3 + 3x^2 + 9x)^{-4}}{-4} + c .$$

$$3. \int \frac{1}{x\sqrt{\ln x}} \, dx = \int (\ln x)^{-\frac{1}{2}} \frac{1}{x} \, dx = \frac{(\ln x)^{\frac{1}{2}}}{\frac{1}{2}} + c .$$

$$4. \int \frac{1}{x \ln \sqrt{x}} \, dx = \int \frac{1}{x^{\frac{1}{2}} \ln x} \, dx = 2 \int \frac{\frac{1}{x}}{\ln x} \, dx = \ln |\ln x| + c .$$

$$5. \int \frac{x-1}{x+1} \, dx = \int \frac{(x+1)-2}{x+1} \, dx = \int \left(\frac{x+1}{x+1} - \frac{2}{x+1} \right) \, dx \\ \int \left(1 - \frac{2}{x+1} \right) \, dx = \int 1 \, dx - 2 \int \frac{1}{x+1} \, dx = x - 2 \ln |x+1| + c .$$

$$6. \text{ Find } g(x) \text{ if } \int [\ln |x|]^2 g(x) \, dx = \frac{2}{3} [\ln |x|]^3 + c$$

$$[\ln |x|]^2 g(x) = \frac{d}{dx} \left(\frac{2}{3} [\ln |x|]^3 + c \right)$$

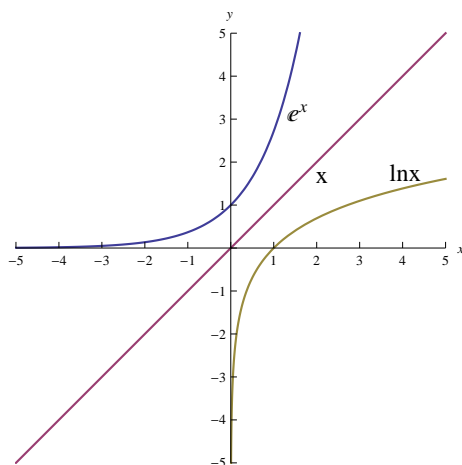
$$[\ln |x|]^2 g(x) = 2 [\ln |x|]^2 \frac{1}{x} .$$

$$\text{Hence } g(x) = \frac{2}{x} .$$

THE NATURAL EXPONENTIAL FUNCTION

Definition (The natural exponential function) :

The natural exponential function is the inverse of the natural logarithmic function , and it is denoted by e^x .



Notes :

1. The domain of the function e^x is \mathbb{R} .
2. The range of the function e^x is the open interval $(0, \infty)$.
3. $e^x > 0$ for every $x \in \mathbb{R}$.
4. $e^0 = 1$.
5. $e \approx 2.71828$ and $\ln(e) = 1$.
6. $\lim_{x \rightarrow \infty} e^x = \infty$.
7. $\lim_{x \rightarrow -\infty} e^x = 0$.
8. $\ln(e^x) = x$ and $e^{\ln x} = x$.

Some properties of the natural exponential function : If $x, y \in \mathbb{R}$ then

1. $e^x e^y = e^{x+y}$.
2. $\frac{e^x}{e^y} = e^{x-y}$.
3. $(e^x)^y = e^{xy}$.

Examples :

1. Find the value of x that satisfies the equation $\ln \frac{1}{x} = 2$?

$$\begin{aligned} \text{Answer : } \ln \frac{1}{x} = 2 &\Rightarrow \ln x^{-1} = 2 \Rightarrow -\ln x = 2 \Rightarrow \ln x = -2 \\ &\Rightarrow e^{\ln x} = e^{-2} \Rightarrow x = e^{-2} = \frac{1}{e^2} . \end{aligned}$$

2. Find the value of x that satisfies the equation $e^{5x+3} = 4$?.

$$\text{Answer : } e^{5x+3} = 4 \Rightarrow \ln e^{5x+3} = \ln 4 \Rightarrow 5x + 3 = \ln 4 \Rightarrow x = \frac{-3 + \ln 4}{5} .$$

3. Simplify $\ln (e^x)^2$?

$$\text{Answer : } \ln (e^x)^2 = \ln (e^{2x}) = 2x .$$

Derivative of the natural exponential function:

- $\frac{d}{dx} e^x = e^x .$
- $\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x) .$

Integration :

- $\int e^x dx = e^x + c .$
- $\int e^{f(x)} f'(x) dx = e^{f(x)} + c .$

Example :

1. Find $f'(x)$ if $f(x) = e^{5x} + \frac{1}{e^x}$

$$\begin{aligned} f(x) &= e^{5x} + \frac{1}{e^x} = e^{5x} + e^{-x} \\ f'(x) &= e^{5x}(5) + e^{-x}(-1) = 5e^{5x} - e^{-x} . \end{aligned}$$

2. $\int \frac{e^{-x}}{(1 - e^{-x})^2} dx = \int (1 - e^{-x})^{-2} e^{-x} dx = \frac{(1 - e^{-x})^{-1}}{-1} + c .$

3. $\int \frac{e^{\frac{3}{x}}}{x^2} dx = -\frac{1}{3} \int e^{\frac{3}{x}} \frac{-3}{x^2} dx = -\frac{1}{3} e^{\frac{3}{x}} + c .$

4. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^{\sqrt{x}} \frac{1}{2\sqrt{x}} dx = 2e^{\sqrt{x}} + c .$

5. $\int \frac{e^{\sin x}}{\sec x} dx = \int e^{\sin x} \cos x dx = e^{\sin x} + c .$

6. $\int_1^e \frac{\sqrt[3]{\ln x}}{x} dx = \int_1^e (\ln x)^{\frac{1}{3}} \frac{1}{x} dx = \left[\frac{(\ln x)^{\frac{4}{3}}}{\frac{4}{3}} \right]_1^e = \frac{3}{4} (\ln e)^{\frac{4}{3}} - \frac{3}{4} (\ln 1)^{\frac{4}{3}} = \frac{3}{4}$

7. Find $g(x)$ if $\int e^{3x^2} g(x) dx = -e^{3x^2} + c$

$$\frac{d}{dx} [-e^{3x^2} + c] = e^{3x^2} g(x)$$

$$-e^{3x^2} (6x) = e^{3x^2} g(x)$$

$$-6xe^{3x^2} = e^{3x^2} g(x)$$

$$\text{Hence } g(x) = -6x$$

8. $\int e^{(x^2+\ln x)} dx = \int e^{x^2} e^{\ln x} dx = \int e^{x^2} x dx = \frac{1}{2} \int e^{x^2} 2x dx = \frac{1}{2} e^{x^2} + c$

THE GENERAL EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Definition (The general exponential function) :

It has the form a^x where $a > 0$ and $a \neq 1$.

Note : $a^x = e^{x \ln a}$.

Derivative of the general exponential function :

1. $\frac{d}{dx} a^x = a^x \ln a$.
2. $\frac{d}{dx} a^{f(x)} = a^{f(x)} f'(x) \ln a$.

Integration :

1. $\int a^x dx = \frac{a^x}{\ln a} + c$.
2. $\int a^{f(x)} f'(x) dx = \frac{a^{f(x)}}{\ln a} + c$.

Definition (The general logarithmic function) :

The general logarithmic function of base a where $a > 0$ and $a \neq 1$ is denoted by $\log_a x$ and it is the inverse function of the general exponential function a^x .

Notes :

1. $\log_a x = y \Leftrightarrow a^y = x$.
2. $\log_a x = \frac{\ln x}{\ln a}$.

Notations :

1. $\log x = \log_{10} x$.
2. $\ln x = \log_e x$.

Derivative of the general logarithmic function :

1. $\frac{d}{dx} \log_a |x| = \frac{1}{x} \frac{1}{\ln a}$.
2. $\frac{d}{dx} \log_a |f(x)| = \frac{f'(x)}{f(x)} \frac{1}{\ln a}$.

Examples :

1. Find the value of x if $\log_2 x = 3$?.

$$\log_2 x = 3 \Leftrightarrow x = 2^3 = 8 .$$

2. Find the value of a if $\log_a 125 = 3$?

$$\log_a 125 = 3 \Leftrightarrow 125 = a^3 \Leftrightarrow a = \sqrt[3]{125} = 5 .$$

3. Find the value of x if $2 \log |x| = \log 2 + \log |3x - 4|$? .

$$2 \log |x| = \log 2 + \log |3x - 4| \Rightarrow \log x^2 = \log |2(3x - 4)|$$

$$\Rightarrow x^2 = 2(3x - 4) \Rightarrow x^2 = 6x - 8 \Rightarrow x^2 - 6x + 8 = 0$$

$$(x - 4)(x - 2) = 0 \Rightarrow x = 4 \text{ or } x = 2 .$$

4. Find y' if $2x = 4^y$?

$$\text{Differentiate both sides : } 2 = 4^y y' \ln 4 \Rightarrow y' = \frac{2}{4^y \ln 4} = \frac{2}{2x \ln 4} = \frac{1}{x \ln 4} .$$

$$\text{Another way : } 2x = 4^y \Rightarrow \ln |2x| = \ln 4^y = y \ln 4 \Rightarrow y = \frac{\ln |2x|}{\ln 4}$$

$$\text{Hence } y' = \frac{1}{\ln 4} \frac{2}{2x} = \frac{1}{x \ln 4} .$$

5. Find $f'(x)$ if $f(x) = 7^{\sqrt[3]{x}}$?.

$$f'(x) = 7^{\sqrt[3]{x}} \frac{1}{3} x^{-\frac{2}{3}} \ln 7 .$$

6. Find $f'(x)$ if $f(x) = \pi^{3x}$?

$$f'(x) = \pi^{3x} (3) \ln \pi = 3\pi^{3x} \ln \pi .$$

7. Find y' if $y = (\sin x)^x$?

$$y = (\sin x)^x \Rightarrow \ln y = \ln (\sin x)^x = x \ln |\sin x|$$

$$\text{Differentiate both sides : } \frac{y'}{y} = \ln |\sin x| + x \frac{\cos x}{\sin x} = \ln |\sin x| + x \cot x$$

$$y' = y [\ln |\sin x| + x \cot x] = (\sin x)^x [\ln |\sin x| + x \cot x]$$

8. Find y' if $y = (1 + x^2)^{2x+1}$?

$$y = (1 + x^2)^{2x+1} \Rightarrow \ln y = \ln (1 + x^2)^{2x+1} = (2x + 1) \ln(1 + x^2)$$

$$\text{Differentiate both sides : } \frac{y'}{y} = 2 \ln(1 + x^2) + (2x + 1) \frac{2x}{1 + x^2}$$

$$y' = y \left[2 \ln(1 + x^2) + \frac{2x(2x + 1)}{1 + x^2} \right] = (1 + x^2)^{2x+1} \left[2 \ln(1 + x^2) + \frac{2x(2x + 1)}{1 + x^2} \right]$$

9. $\int x^2 6^{x^3} dx = \frac{1}{3 \ln 6} \int 6^{x^3} (3x^2) \ln 6 dx = \frac{6^{x^3}}{3 \ln 6} + c .$

10. $\int \frac{2^x}{2^x + 1} dx = \frac{1}{\ln 2} \int \frac{2^x \ln 2}{2^x + 1} dx = \frac{\ln(2^x + 1)}{\ln 2} + c .$

11. $\int \frac{3^{-\cot x}}{\sin^2 x} dx = \frac{1}{\ln 3} \int 3^{-\cot x} \csc^2 x \ln 3 dx = \frac{3^{-\cot x}}{\ln 3} + c$

12. $\int 2^{x \ln x} (1 + \ln |x|) dx = \frac{1}{\ln 2} \int 2^{x \ln x} (1 + \ln |x|) \ln 2 dx = \frac{2^{x \ln x}}{\ln 2} + c$

13. $\int 4^x 5^{4^x} dx = \frac{1}{\ln 4 \ln 5} \int 5^{4^x} 4^x \ln 4 \ln 5 dx = \frac{5^{4^x}}{\ln 4 \ln 5} + c$

$$\begin{aligned}
14. \int 3^x (1 + \sin 3^x) dx &= \int (3^x + 3^x \sin 3^x) dx = \int 3^x dx + \int 3^x \sin 3^x dx \\
&= \frac{1}{\ln 3} \int 3^x \ln 3 dx + \frac{1}{\ln 3} \int \sin(3^x) 3^x \ln 3 dx = \frac{3^x}{\ln 3} - \frac{\cos 3^x}{\ln 3} + c
\end{aligned}$$

Exercises :

1. Find $f'(x)$ if $f(x) = (x^2 + 1)^x$?

2. Evaluate $\int \frac{3^{\sqrt{x}}}{\sqrt{x}} dx$?

THE INVERSE TRIGONOMETRIC FUNCTIONS

Definitions :

1. The inverse sine function is denoted by \sin^{-1} and it is defined as $y = \sin^{-1} x \Leftrightarrow x = \sin y$, where $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The domain of the inverse sine function is $[-1, 1]$

The range of the inverse sine function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

2. The inverse cosine function is denoted by \cos^{-1} and it is defined as $y = \cos^{-1} x \Leftrightarrow x = \cos y$, where $x \in [-1, 1]$ and $y \in [0, \pi]$.

The domain of the inverse cosine function is $[-1, 1]$

The range of the inverse cosine function is $[0, \pi]$.

3. The inverse tangent function is denoted by \tan^{-1} and it is defined as $y = \tan^{-1} x \Leftrightarrow x = \tan y$, where $x \in \mathbb{R}$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The domain of the inverse tangent function is \mathbb{R}

The range of the inverse tangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

4. The inverse cotangent function is denoted by \cot^{-1} and it is defined as $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$, where $x \in \mathbb{R}$.

The domain of the inverse cotangent function is \mathbb{R}

The range of the inverse cotangent function is $(0, \pi)$.

5. The inverse secant function is denoted by \sec^{-1} and it is defined as $y = \sec^{-1} x \Leftrightarrow x = \sec y$, where $y \in \left[0, \frac{\pi}{2}\right)$ if $x \geq 1$, and $y \in \left[\pi, \frac{3\pi}{2}\right)$ if $x \leq -1$.

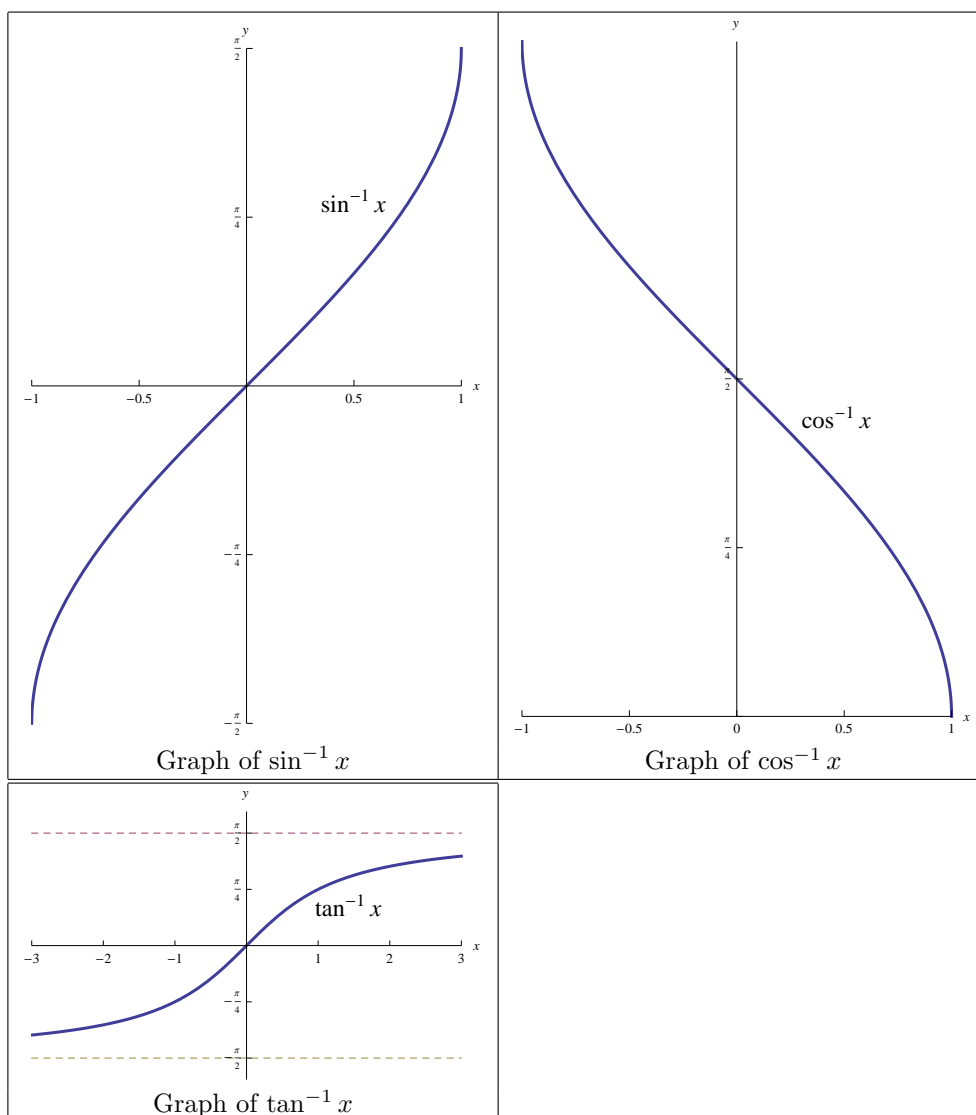
The domain of the inverse secant function is $(-\infty, -1] \cup [1, \infty)$

The range of the inverse secant function is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.

6. The inverse cosecant function is denoted by \csc^{-1} and it is defined as $\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$ where $|x| \geq 1$

The domain of the inverse cosecant function is $(-\infty, -1] \cup [1, \infty)$

The range of the inverse cosecant function is $\left(-\pi, -\frac{\pi}{2}\right] \cup \left(0, \frac{\pi}{2}\right]$.



Derivatives of the inverse trigonometric functions :

1. $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, where $|x| < 1$.
2. $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$, where $|x| < 1$.
3. $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$.
4. $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$.
5. $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$, where $|x| > 1$.

$$6. \frac{d}{dx} \csc^{-1} x = \frac{-1}{x\sqrt{x^2-1}}, \text{ where } |x| > 1.$$

Integration :

1. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c, (|x| < a)$
 $\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}} dx = \sin^{-1} \left(\frac{f(x)}{a} \right) + c, (|f(x)| < a)$
2. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$
 $\int \frac{f'(x)}{a^2+[f(x)]^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{f(x)}{a} \right) + c$
3. $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} \sec^{-1} \left(\frac{x}{a} \right) + c, (|x| > a)$
 $\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2-a^2}} dx = \frac{1}{a} \sec^{-1} \left(\frac{f(x)}{a} \right) + c, (|f(x)| > a)$

Examples :

$$1. \int \frac{x^2}{5+x^6} dx = \frac{1}{3} \int \frac{3x^2}{(\sqrt{5})^2+(x^3)^2} dx = \frac{1}{3} \frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{x^3}{\sqrt{5}} \right) + c.$$

Here $a = \sqrt{5}$, $f(x) = x^3$ and $f'(x) = 3x^2$.

$$2. \int \frac{3x}{\sqrt{9-x^4}} dx = \frac{3}{2} \int \frac{2x}{\sqrt{(3)^2-(x^2)^2}} dx = \frac{3}{2} \sin^{-1} \left(\frac{x^2}{3} \right) + c.$$

Here $a = 3$, $f(x) = x^2$ and $f'(x) = 2x$.

$$3. \int \frac{3x}{\sqrt{9-x^2}} dx = \frac{3}{-2} \int (9-x^2)^{-\frac{1}{2}} (-2x) dx = -\frac{3}{2} \frac{(9-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c.$$

$$4. \int \frac{1}{x\sqrt{1-(\ln x)^2}} dx = \int \frac{\left(\frac{1}{x}\right)}{\sqrt{(1)^2-(\ln x)^2}} dx = \sin^{-1}(\ln x) + c.$$

Here $a = 1$, $f(x) = \ln x$ and $f'(x) = \frac{1}{x}$.

$$5. \int \frac{1}{1+3x^2} dx = \frac{1}{\sqrt{3}} \int \frac{\sqrt{3}}{(1)^2+(\sqrt{3}x)^2} dx = \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}x) + c.$$

Here $a = 1$, $f(x) = \sqrt{3}x$ and $f'(x) = \sqrt{3}$.

$$6. \int \frac{e^{2x}}{e^{4x}+16} dx = \frac{1}{2} \int \frac{2e^{2x}}{(4)^2+(e^{2x})^2} dx = \frac{1}{2} \frac{1}{4} \tan^{-1} \left(\frac{e^{2x}}{4} \right) + c.$$

Here $a = 4$, $f(x) = e^{2x}$ and $f'(x) = 2e^{2x}$.

$$7. \int \frac{1}{\sqrt{e^{2x} - 36}} dx = \int \frac{e^x}{e^x \sqrt{(e^x)^2 - (6)^2}} dx = \frac{1}{6} \sec^{-1} \left(\frac{e^x}{6} \right) + c .$$

Here $a = 6$, $f(x) = e^2$ and $f'(x) = e^x$.

$$8. \int \frac{\sin x}{\sqrt{25 - \cos^2 x}} dx = - \int \frac{-\sin x}{\sqrt{(5)^2 - (\cos x)^2}} dx = - \sin^{-1} \left(\frac{\cos x}{5} \right) + c .$$

Here $a = 5$, $f(x) = \cos x$ and $f'(x) = -\sin x$.

$$9. \int \frac{2^x}{\sqrt{4 - 4^x}} dx = \frac{1}{\ln 2} \int \frac{2^x \ln 2}{\sqrt{(2)^2 - (2^x)^2}} dx = \frac{1}{\ln 2} \sin^{-1} \left(\frac{2^x}{2} \right) + c .$$

Here $a = 2$, $f(x) = 2^x$ and $f'(x) = 2^x \ln 2$.

$$10. \int \frac{1}{x^2 + 6x + 25} dx = \int \frac{1}{(x^2 + 6x + 9) + 16} dx = \int \frac{1}{(x + 3)^2 + (4)^2} dx \\ = \frac{1}{4} \tan^{-1} \left(\frac{x + 3}{4} \right) + c .$$

Here $a = 4$, $f(x) = x + 3$ and $f'(x) = 1$.

$$11. \int \frac{x + 2}{\sqrt{4 - x^2}} dx = \int \left(\frac{x}{\sqrt{4 - x^2}} + \frac{2}{\sqrt{4 - x^2}} \right) dx \\ = \frac{1}{-2} \int (4 - x^2)^{-\frac{1}{2}} (-2x) dx + 2 \int \frac{1}{\sqrt{(2)^2 - (x)^2}} dx \\ = -\frac{1}{2} \frac{(4 - x^2)^{\frac{1}{2}}}{\frac{1}{2}} + 2 \sin^{-1} \left(\frac{x}{2} \right) + c .$$

$$12. \int \frac{x + \tan^{-1} x}{1 + x^2} dx = \int \left(\frac{x}{1 + x^2} + \frac{\tan^{-1} x}{1 + x^2} \right) dx \\ = \frac{1}{2} \int \frac{2x}{1 + x^2} dx + \int (\tan^{-1} x) \frac{1}{1 + x^2} dx \\ = \frac{1}{2} \ln(1 + x^2) + \frac{(\tan^{-1} x)^2}{2} + c .$$

Exercises : Solve the following integrals :

$$1. \int \frac{x + \sin^{-1} x}{\sqrt{1 - x^2}} dx .$$

$$2. \int \frac{x + 1}{x^2 + 1} dx$$

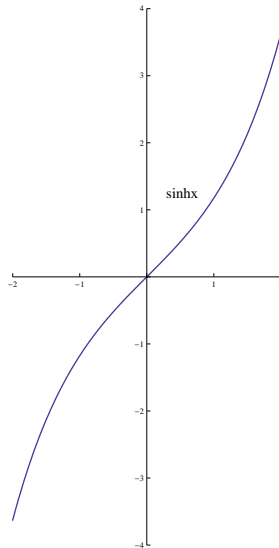
HYPERBOLIC FUNCTIONS

Definition (The hyperbolic sine function):

It is denoted by $\sinh x$ and it is defined as $\sinh x = \frac{e^x - e^{-x}}{2}$.

Notes :

1. The domain of $\sinh x$ is \mathbb{R} .
2. The range of $\sinh x$ is \mathbb{R} .
3. It is an odd function and $\sinh(0) = 0$.
4. The graph of $\sinh x$

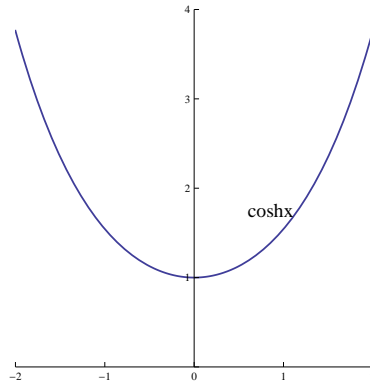


Definition (The hyperbolic cosine function):

It is denoted by $\cosh x$ and it is defined as $\cosh x = \frac{e^x + e^{-x}}{2}$.

Notes :

1. The domain of $\cosh x$ is \mathbb{R} .
2. The range of $\cosh x$ is $[1, \infty]$.
3. It is an even function and $\cosh(0) = 1$.
4. The graph of $\cosh x$



Definitions :

1. The hyperbolic tangent function is denoted by $\tanh x$ and it is defined as $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ for every $x \in \mathbb{R}$.
2. The hyperbolic cotangent function is denoted by $\coth x$ and it is defined as $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ for every $x \in \mathbb{R} - \{0\}$.
3. The hyperbolic secant function is denoted by $\operatorname{sech} x$ and it is defined as $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$ for every $x \in \mathbb{R}$.
4. The hyperbolic cosecant function is denoted by $\operatorname{csch} x$ and it is defined as $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$ for every $x \in \mathbb{R} - \{0\}$.

Notes :

1. $\cosh^2 x - \sinh^2 x = 1$ for every $x \in \mathbb{R}$.
2. $1 - \tanh^2 x = \operatorname{sech}^2 x$ for every $x \in \mathbb{R}$.
3. $\coth^2 x - 1 = \operatorname{csch}^2 x$ for every $x \in \mathbb{R} - \{0\}$.

Derivatives of the hyperbolic functions :

1. $\frac{d}{dx} \sinh x = \cosh x$
 $\frac{d}{dx} \sinh(f(x)) = \cosh(f(x)) f'(x)$
2. $\frac{d}{dx} \cosh x = \sinh x$
 $\frac{d}{dx} \cosh(f(x)) = \sinh(f(x)) f'(x)$

3. $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$
 $\frac{d}{dx} \tanh(f(x)) = \operatorname{sech}^2(f(x)) f'(x)$
4. $\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$
 $\frac{d}{dx} \coth(f(x)) = -\operatorname{csch}^2(f(x)) f'(x)$
5. $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$
 $\frac{d}{dx} \operatorname{sech}(f(x)) = -\operatorname{sech}(f(x)) \tanh(f(x)) f'(x)$
6. $\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$
 $\frac{d}{dx} \operatorname{csch}(f(x)) = -\operatorname{csch}(f(x)) \coth(f(x)) f'(x)$

Examples :

1. Find the value of $f(0)$ if $f(x) = \ln [\cosh(3x)]$?
 $f(0) = \ln [\cosh(0)] = \ln(1) = 0$.
2. Find the value of $f'(0)$ if $f(x) = \ln |1 + \sinh x|$?
 $f'(x) = \frac{\cosh x}{1 + \sinh x} \Rightarrow f'(0) = \frac{\cosh(0)}{1 + \sinh(0)} = \frac{1}{1 + 0} = 1$.
3. Find $f'(x)$ if $f(x) = e^{\sinh x}$?
 $f'(x) = e^{\sinh x} \cosh x$.
4. Find $f'(x)$ if $f(x) = \operatorname{sech}(1 + \sqrt{x})$?
 $f'(x) = -\operatorname{sech}(1 + \sqrt{x}) \tanh(1 + \sqrt{x}) \frac{1}{2\sqrt{x}}$.
5. Find $f'(x)$ if $f(x) = \tan^{-1}(\sinh x)$?
 $f'(x) = \frac{\cosh x}{1 + (\sinh x)^2} = \frac{\cosh x}{\cosh^2 x} = \frac{1}{\cosh x} = \operatorname{sech} x$.
6. Find $f'(x)$ if $f(x) = \ln |\sinh(1 - x^2)|$?
 $f'(x) = \frac{\cosh(1 - x^2) (-2x)}{\sinh(1 - x^2)} = -2x \coth(1 - x^2)$.
7. Find $f'(x)$ if $f(x) = x^{\cosh x}$?
 $f(x) = x^{\cosh x} \Rightarrow \ln |f(x)| = \ln |x^{\cosh x}| = \cosh x \ln |x|$

Differentiate both sides

$$\frac{f'(x)}{f(x)} = \sinh x \ln |x| + \cosh x \left(\frac{1}{x} \right)$$

$$f'(x) = f(x) \left[\sinh x \ln |x| + \frac{\cosh x}{x} \right]$$

$$f'(x) = x^{\cosh x} \left[\sinh x \ln |x| + \frac{\cosh x}{x} \right].$$

Integration :

$$1. \int \sinh x \, dx = \cosh x + c$$

$$\int \sinh (f(x)) f'(x) \, dx = \cosh (f(x)) + c$$

$$2. \int \cosh x \, dx = \sinh x + c$$

$$\int \cosh (f(x)) f'(x) \, dx = \sinh (f(x)) + c$$

$$3. \int \operatorname{sech}^2 x \, dx = \tanh x + c$$

$$\int \operatorname{sech}^2 (f(x)) f'(x) \, dx = \tanh (f(x)) + c$$

$$4. \int \operatorname{csch}^2 x \, dx = -\operatorname{coth} x + c$$

$$\int \operatorname{csch}^2 (f(x)) f'(x) \, dx = -\operatorname{coth} (f(x)) + c$$

$$5. \int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$$

$$\int \operatorname{sech} (f(x)) \tanh (f(x)) f'(x) \, dx = -\operatorname{sech} (f(x)) + c$$

$$6. \int \operatorname{csch} x \operatorname{coth} x \, dx = -\operatorname{csch} x + c$$

$$\int \operatorname{csch} (f(x)) \operatorname{coth} (f(x)) f'(x) \, dx = -\operatorname{csch} (f(x)) + c$$

$$7. \int \tanh x \, dx = \ln |\cosh x| + c$$

$$\int \tanh (f(x)) f'(x) \, dx = \ln |\cosh (f(x))| + c$$

$$8. \int \operatorname{coth} x \, dx = \ln |\sinh x| + c$$

$$\int \coth(f(x)) f'(x) dx = \ln |\sinh(f(x))| + c$$

Examples :

$$1. \int x^2 \cosh x^3 dx = \frac{1}{3} \int \cosh x^3 (3x^2) dx = \frac{1}{3} \sinh x^3 + c .$$

$$2. \int \frac{\operatorname{csch}\left(\frac{1}{x}\right) \coth\left(\frac{1}{x}\right)}{x^2} dx = \int -\operatorname{csch}\left(\frac{1}{x}\right) \coth\left(\frac{1}{x}\right) \left(\frac{-1}{x^2}\right) dx \\ = \operatorname{csch}\left(\frac{1}{x}\right) + c .$$

$$3. \int (e^x - e^{-x}) \operatorname{sech}^2(e^x + e^{-x}) dx = \tanh(e^x + e^{-x}) + c .$$

$$4. \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \ln |e^x + e^{-x}| + c .$$

$$5. \int \frac{\sinh x}{1 + \sinh^2 x} dx = \int \frac{\sinh x}{\cosh^2 x} dx = \int \frac{1}{\cosh x} \frac{\sinh x}{\cosh x} dx \\ = \int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + c .$$

$$6. \int \frac{\sinh x}{1 + \cosh x} dx = \ln(1 + \cosh x) + c .$$

$$7. \int \frac{\sinh x}{1 + \cosh^2 x} dx = \int \frac{\sinh x}{(1)^2 + (\cosh x)^2} dx = \tan^{-1}(\cosh x) + c .$$

$$8. \int \frac{1}{\operatorname{sech} x \sqrt{4 - \sinh^2 x}} dx = \int \frac{\cosh x}{\sqrt{(2)^2 - (\sinh x)^2}} dx = \sin^{-1}\left(\frac{\sinh x}{2}\right) + c$$

Exercises : Solve the following :

$$1. \int \cosh 4x dx$$

$$2. \int \frac{\sinh \sqrt{x}}{\sqrt{x}} dx$$

THE INVERSE HYPERBOLIC FUNCTIONS

Definitions :

1. The inverse hyperbolic sine function is denoted by \sinh^{-1} and it is defined as $y = \sinh^{-1} x \Leftrightarrow x = \sinh y$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.
2. The inverse hyperbolic cosine function is denoted by \cosh^{-1} and it is defined as $y = \cosh^{-1} x \Leftrightarrow x = \cosh y$, where $x \in [1, \infty)$ and $y \in [0, \infty)$.
3. The inverse hyperbolic tangent function is denoted by \tanh^{-1} and it is defined as $y = \tanh^{-1} x \Leftrightarrow x = \tanh y$, where $x \in [-1, 1]$ and $y \in \mathbb{R}$.
4. The inverse hyperbolic cotangent function is denoted by \coth^{-1} and it is defined as $y = \coth^{-1} x \Leftrightarrow x = \coth y$, where $|x| > 1$ and $y \in \mathbb{R}$.
5. The inverse hyperbolic secant function is denoted by sech^{-1} and it is defined as $y = \operatorname{sech}^{-1} x \Leftrightarrow x = \operatorname{sech} y$, where $x \in (0, 1]$ and $y \in [0, \infty)$.
6. The inverse hyperbolic cosecant function is denoted by csch^{-1} and it is defined as $y = \operatorname{csch}^{-1} x \Leftrightarrow x = \operatorname{csch} y$, where $x \in \mathbb{R} - \{0\}$ and $y \in \mathbb{R} - \{0\}$.

Derivatives of the inverse hyperbolic functions :

1.
$$\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}} .$$
$$\frac{d}{dx} \sinh^{-1}(f(x)) = \frac{f'(x)}{\sqrt{1+(f(x))^2}} .$$
2.
$$\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}} , \text{ where } x > 1.$$
$$\frac{d}{dx} \cosh^{-1}(f(x)) = \frac{f'(x)}{\sqrt{(f(x))^2-1}} , \text{ where } f(x) > 1.$$
3.
$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2} , \text{ where } |x| < 1.$$
$$\frac{d}{dx} \tanh^{-1}(f(x)) = \frac{f'(x)}{1-(f(x))^2} , \text{ where } |f(x)| < 1.$$
4.
$$\frac{d}{dx} \coth^{-1} x = \frac{1}{1-x^2} , \text{ where } |x| > 1.$$
$$\frac{d}{dx} \coth^{-1}(f(x)) = \frac{f'(x)}{1-(f(x))^2} , \text{ where } |f(x)| > 1.$$
5.
$$\frac{d}{dx} \operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1-x^2}} , \text{ where } 0 < x < 1.$$
$$\frac{d}{dx} \operatorname{sech}^{-1}(f(x)) = \frac{-f'(x)}{f(x)\sqrt{1-(f(x))^2}} , \text{ where } 0 < f(x) < 1.$$

$$6. \frac{d}{dx} \operatorname{csch}^{-1} x = \frac{-1}{|x|\sqrt{1+x^2}}, \text{ where } x \neq 0.$$

$$\frac{d}{dx} \operatorname{csch}^{-1}(f(x)) = \frac{-f'(x)}{|f(x)|\sqrt{1+(f(x))^2}}, \text{ where } f(x) \neq 0.$$

Examples :

1. Find $f'(x)$ if $f(x) = \tanh^{-1} 3x$?

$$f'(x) = \frac{3}{1-(3x)^2} = \frac{3}{1-9x^2}.$$

2. Find $f'(x)$ if $f(x) = \sinh^{-1} \sqrt{x}$?

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1+(\sqrt{x})^2}} = \frac{1}{2\sqrt{x}\sqrt{1+x}}.$$

3. Find $f'(x)$ if $f(x) = \operatorname{sech}^{-1}(\cos 2x)$?

$$f'(x) = \frac{-(-2 \sin 2x)}{\cos 2x \sqrt{1-(\cos 2x)^2}} = \frac{2 \sin 2x}{\cos 2x \sqrt{1-\cos^2 2x}}.$$

Integration :

1. $\int \frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1} \left(\frac{x}{a} \right) + c$

$$\int \frac{f'(x)}{\sqrt{a^2+[f(x)]^2}} dx = \sinh^{-1} \left(\frac{f(x)}{a} \right) + c$$

2. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1} \left(\frac{x}{a} \right) + c, (x > a)$

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2-a^2}} dx = \cosh^{-1} \left(\frac{f(x)}{a} \right) + c, (f(x) > a)$$

3. $\int \frac{1}{a^2-x^2} dx = \frac{1}{a} \tanh^{-1} \left(\frac{x}{a} \right) + c, (|x| < a)$

$$\int \frac{f'(x)}{a^2-[f(x)]^2} dx = \frac{1}{a} \tanh^{-1} \left(\frac{f(x)}{a} \right) + c, (|f(x)| < a)$$

4. $\int \frac{1}{x\sqrt{a^2-x^2}} dx = -\frac{1}{a} \operatorname{sech}^{-1} \left(\frac{x}{a} \right) + c, (0 < x < a)$

$$\int \frac{f'(x)}{f(x)\sqrt{a^2-[f(x)]^2}} dx = -\frac{1}{a} \operatorname{sech}^{-1} \left(\frac{f(x)}{a} \right) + c, (0 < f(x) < a)$$

5. $\int \frac{1}{x\sqrt{x^2+a^2}} dx = -\frac{1}{a} \operatorname{csch}^{-1} \left(\frac{x}{a} \right) + c, (x \neq 0)$

$$\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 + a^2}} dx = -\frac{1}{a} \operatorname{csch}^{-1}\left(\frac{f(x)}{a}\right) + c, \quad (f(x) \neq 0)$$

Examples :

$$1. \int \frac{e^x}{1 - e^{2x}} dx = \int \frac{e^x}{(1)^2 - (e^x)^2} dx = \tanh^{-1}(e^x) + c.$$

$$2. \int \frac{e^x}{\sqrt{4e^{2x} + 9}} dx = \frac{1}{2} \int \frac{2e^x}{\sqrt{(2e^x)^2 + (3)^2}} dx = \frac{1}{2} \sinh^{-1}\left(\frac{2e^x}{3}\right) + c.$$

$$3. \int \frac{1}{\sqrt{x}\sqrt{4+x}} dx = 2 \int \frac{\frac{1}{2\sqrt{x}}}{\sqrt{(2)^2 + (\sqrt{x})^2}} dx = 2 \sinh^{-1}\left(\frac{\sqrt{x}}{2}\right) + c.$$

$$4. \int \frac{1}{\sqrt{16 - e^{2x}}} dx = \int \frac{e^x}{e^x \sqrt{(4)^2 - (e^x)^2}} dx = -\frac{1}{4} \operatorname{sech}^{-1}\left(\frac{e^x}{4}\right) + c.$$

$$5. \int \frac{1}{\sqrt{1 + e^{2x}}} dx = \int \frac{e^x}{e^x \sqrt{(1)^2 + (e^x)^2}} dx = -\operatorname{csch}^{-1}(e^x) + c.$$

$$6. \int \frac{1}{\sqrt{x^2 + 2x - 8}} dx = \int \frac{1}{\sqrt{(x^2 + 2x + 1) - 9}} dx = \int \frac{1}{\sqrt{(x+1)^2 - (3)^2}} dx \\ = \cosh^{-1}\left(\frac{x+1}{3}\right) + c.$$

$$7. \int \frac{1}{(x-1)\sqrt{-x^2 + 2x + 3}} dx = \int \frac{1}{(x-1)\sqrt{-(x^2 - 2x + 1) + 4}} dx \\ = \int \frac{1}{(x-1)\sqrt{(2)^2 - (x-1)^2}} dx = -\frac{1}{2} \operatorname{sech}^{-1}\left(\frac{x-1}{2}\right) + c.$$

INDETERMINATE FORMS

Theorem (L'Hôpital's Rule) :

Suppose that f and g are differentiable on the interval (a, b) , except possibly at a point $c \in (a, b)$ and that $g'(x) \neq 0$ on (a, b) , except possibly at c .

Suppose further that $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and that

$$\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = L \text{ (or } \pm\infty \text{)}. \text{ Then, } \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}.$$

Remark :

The conclusion of the theorem also holds if $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ is replaced with $\lim_{x \rightarrow c^+} \frac{f(x)}{g(x)}$,

$\lim_{x \rightarrow c^-} \frac{f(x)}{g(x)}$, $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ or $\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)}$. (In each case, we must make appropriate adjustment of the hypothesis.)

Types of indeterminate forms :

1. $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
2. $\infty - \infty$ or $-\infty + \infty$.
3. $0 \cdot \infty$ or $0(-\infty)$.
4. 0^0 , 1^∞ , $1^{-\infty}$ or ∞^0 .

Examples :

$$1. \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{\ln x} \quad \left(\frac{0}{0} \right)$$

Apply L'Hôpital's rule

$$\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{\ln x} = \lim_{x \rightarrow 1} \frac{\left(\frac{1}{2\sqrt{x}} \right)}{\left(\frac{1}{x} \right)} = \lim_{x \rightarrow 1} \frac{x}{2\sqrt{x}} = \frac{1}{2}.$$

$$2. \lim_{x \rightarrow 0} \frac{\sin x \sqrt{1 - \sin x}}{x} \quad \left(\frac{0}{0} \right)$$

$$\lim_{x \rightarrow 0} \frac{\sin x \sqrt{1 - \sin x}}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \sqrt{1 - \sin x} = 1 \sqrt{1 - 0} = 1.$$

$$3. \lim_{x \rightarrow 0} \frac{\int_0^x \sqrt{1 + \sin t} dt}{x} \quad \left(\frac{0}{0} \right)$$

Apply L'Hôpital's rule

$$\lim_{x \rightarrow 0} \frac{\int_0^x \sqrt{1 + \sin t} dt}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin x}}{1} = \frac{1 + 0}{1} = 1.$$

$$4. \lim_{x \rightarrow 1} \frac{\tan^{-1} x - \frac{\pi}{4}}{x - 1} \quad \left(\frac{0}{0} \right)$$

Apply L'Hôpital's rule

$$\lim_{x \rightarrow 1} \frac{\tan^{-1} x - \frac{\pi}{4}}{x - 1} = \lim_{x \rightarrow 1} \frac{\left(\frac{1}{1 + x^2} \right)}{1} = \lim_{x \rightarrow 1} \frac{1}{1 + x^2} = \frac{1}{1 + 1} = \frac{1}{2}.$$

$$5. \lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} \quad \left(\frac{0}{0} \right)$$

Apply L'Hôpital's rule

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{\tan^2 x}{3x^2} \\ &= \frac{1}{3} \lim_{x \rightarrow 0} \left(\frac{\tan x}{x} \right)^2 = \frac{1}{3} (1)^2 = \frac{1}{3}. \end{aligned}$$

$$6. \lim_{x \rightarrow \infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty} \right)$$

Apply L'Hôpital's rule

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x} \right)}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

$$7. \lim_{x \rightarrow \infty} \frac{x + e^x}{1 + e^{3x}} \quad \left(\frac{\infty}{\infty} \right)$$

Apply L'Hôpital's rule

$$\lim_{x \rightarrow \infty} \frac{x + e^x}{1 + e^{3x}} = \lim_{x \rightarrow \infty} \frac{1 + e^x}{3e^{3x}} \quad \left(\frac{\infty}{\infty} \right)$$

Apply L'Hôpital's rule

$$\lim_{x \rightarrow \infty} \frac{1 + e^x}{3e^{3x}} = \lim_{x \rightarrow \infty} \frac{e^x}{9e^{3x}} = \lim_{x \rightarrow \infty} \frac{1}{9e^{2x}} = 0.$$

$$8. \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{2 - \sec x}{3 \tan x} \quad \left(\frac{-\infty}{\infty} \right)$$

Apply L'Hôpital's rule

$$\begin{aligned} \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{2 - \sec x}{3 \tan x} &= \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{-\sec x \tan x}{3 \sec^2 x} \\ &= \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{-\tan x}{3 \sec x} = \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{-\sin x}{3} = -\frac{1}{3}. \end{aligned}$$

$$9. \lim_{x \rightarrow 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1} \right) \quad (\infty - \infty)$$

$$\lim_{x \rightarrow 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1} \right) = \lim_{x \rightarrow 1^+} \frac{3(x-1) - 2 \ln x}{(x-1) \ln x} \quad \left(\frac{0}{0} \right)$$

Apply L'Hôpital's rule

$$\lim_{x \rightarrow 1^+} \frac{3(x-1) - 2 \ln x}{(x-1) \ln x} = \lim_{x \rightarrow 1^+} \frac{3 - \frac{2}{x}}{\ln x + (x-1) \frac{1}{x}} = \lim_{x \rightarrow 1^+} \frac{3 - \frac{2}{x}}{\ln x + 1 - \frac{1}{x}} = \infty$$

Note that $3 - \frac{2}{x} \rightarrow 1$ and $\ln x + 1 - \frac{1}{x} \rightarrow 0^+$ as $x \rightarrow 1^+$

$$10. \lim_{x \rightarrow \infty} (x^2 - 1)e^{-x^2} \quad (0 \infty)$$

$$\lim_{x \rightarrow \infty} (x^2 - 1)e^{-x^2} = \lim_{x \rightarrow \infty} \frac{x^2 - 1}{e^{x^2}} \quad \left(\frac{\infty}{\infty} \right)$$

Apply L'Hôpital's rule

$$\lim_{x \rightarrow \infty} \frac{x^2 - 1}{e^{x^2}} = \lim_{x \rightarrow \infty} \frac{2x}{2x e^{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{e^{x^2}} = 0$$

$$11. \lim_{x \rightarrow 0^+} x^x \quad (0^0)$$

Put $y = x^x \Leftrightarrow \ln y = \ln x^x = x \ln x$

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} x \ln x \quad (0(-\infty))$$

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-1}} \quad \left(\frac{-\infty}{\infty} \right)$$

Apply L'Hôpital's rule

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-1}} = \lim_{x \rightarrow 0^+} \frac{\left(\frac{1}{x} \right)}{-x^{-2}} = \lim_{x \rightarrow 0^+} (-x) = 0$$

Therefore, $\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} y = e^0 = 1$.

$$12. \lim_{x \rightarrow \infty} (1 + e^{2x})^{\frac{1}{x}} \quad (\infty^0)$$

$$\text{Put } y = (1 + e^{2x})^{\frac{1}{x}} \Leftrightarrow \ln y = \frac{1}{x} \ln(1 + e^{2x}) = \frac{\ln(1 + e^{2x})}{x}$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln(1 + e^{2x})}{x} \quad \left(\frac{\infty}{\infty} \right)$$

Apply L'Hôpital's rule

$$\lim_{x \rightarrow \infty} \frac{\ln(1 + e^{2x})}{x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{2e^{2x}}{1 + e^{2x}} \right)}{1} = \lim_{x \rightarrow \infty} \frac{2e^{2x}}{1 + e^{2x}} \quad \left(\frac{\infty}{\infty} \right)$$

Apply L'Hôpital's rule

$$\lim_{x \rightarrow \infty} \frac{4e^{2x}}{2e^{2x}} = 2$$

Therefore, $\lim_{x \rightarrow \infty} (1 + e^{2x})^{\frac{1}{x}} = \lim_{x \rightarrow \infty} y = e^2$.

13. $\lim_{x \rightarrow \infty} \left(1 + \frac{\ln 3}{x}\right)^x \quad (1^\infty)$

Put $y = \left(1 + \frac{\ln 3}{x}\right)^x \Leftrightarrow \ln y = x \ln \left(1 + \frac{\ln 3}{x}\right)$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{\ln 3}{x}\right) \quad (0 \cdot \infty)$$

$$\lim_{x \rightarrow \infty} x \ln \left(1 + \frac{\ln 3}{x}\right) = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{\ln 3}{x}\right)}{x^{-1}} \quad \left(\frac{0}{0}\right)$$

Apply L'Hôpital's rule

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{\ln 3}{x}\right)}{x^{-1}} &= \lim_{x \rightarrow \infty} \frac{\left(\frac{-\ln 3x^{-2}}{1 + \frac{\ln 3}{x}}\right)}{-x^{-2}} \\ &= \lim_{x \rightarrow \infty} \frac{\ln 3}{\left(1 + \frac{\ln 3}{x}\right)} = \frac{\ln 3}{1 + 0} = \ln 3 \end{aligned}$$

Therefore, $\lim_{x \rightarrow \infty} \left(1 + \frac{\ln 3}{x}\right)^x = \lim_{x \rightarrow \infty} y = e^{\ln 3} = 3$

NOTE : $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x = e^a$ where $a \neq 0$

14. $\lim_{x \rightarrow 0^+} (2x + 1)^{\cot x} \quad (1^\infty)$

Put $y = (2x + 1)^{\cot x} \Leftrightarrow \ln y = \cot x \ln(2x + 1) = \frac{\ln(2x + 1)}{\tan x}$

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln(2x + 1)}{\tan x} \quad \left(\frac{0}{0}\right)$$

Apply L'Hôpital's rule

$$\lim_{x \rightarrow 0^+} \frac{\ln(2x + 1)}{\tan x} = \lim_{x \rightarrow 0^+} \frac{\left(\frac{2}{2x + 1}\right)}{\sec^2 x} = \lim_{x \rightarrow 0^+} \frac{2}{(2x + 1) \sec^2 x} = \frac{2}{2(1)^2} = 2$$

Therefore, $\lim_{x \rightarrow 0^+} (2x + 1)^{\cot x} = \lim_{x \rightarrow 0^+} y = e^2$.

Exercises : Evaluate the following limits

1. $\lim_{x \rightarrow \infty} \frac{4e^x}{x^2}$.

2. $\lim_{x \rightarrow \infty} \frac{e^{2x} - 1}{x}$.

3. $\lim_{x \rightarrow \infty} e^{-x} \sqrt{x}$.

4. $\lim_{x \rightarrow \infty} (1 + 4x)^{\frac{1}{x^2}}$.

5. $\lim_{x \rightarrow 0} \frac{x - \tan x}{1 - \cos x}$.

6. $\lim_{x \rightarrow 0^+} (\sec x + \tan x)^{\csc x}$.

INTEGRATION BY PARTS

It is used to solve integration of a product of two functions using the formula

$$\int u dv = u v - \int v du .$$

Examples :

1. $\int x e^x dx$

$$\begin{aligned} u &= x & dv &= e^x dx \\ du &= dx & v &= e^x \end{aligned}$$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + c = (x - 1)e^x + c .$$

2. $\int x \sec^2 x dx$

$$\begin{aligned} u &= x & dv &= \sec^2 x dx \\ du &= dx & v &= \tan x \end{aligned}$$

$$\int x \sec^2 x dx = x \tan x - \int \tan x dx = x \tan x - \ln |\sec x| + c .$$

3. $\int_0^\pi x \sin x dx$

$$\begin{aligned} u &= x & dv &= \sin x dx \\ du &= dx & v &= -\cos x \end{aligned}$$

$$\int_0^\pi x \sin x dx = [-x \cos x]_0^\pi + \int_0^\pi \cos x dx = [-x \cos x]_0^\pi + [\sin x]_0^\pi .$$

$$= [(-\pi \cos \pi) - (-(0) \cos 0)] + [\sin \pi - \sin 0] = [-\pi(-1) - 0] + [0 - 0] = \pi$$

4. $\int x^2 \sin x dx$

$$\begin{aligned} u &= x^2 & dv &= \sin x dx \\ du &= 2x dx & v &= -\cos x \end{aligned}$$

$$\int x^2 \sin x dx = -x^2 \cos x + \int 2x \cos x dx$$

Now to solve $\int 2x \cos x dx$

$$\begin{aligned} u &= 2x & dv &= \cos x dx \\ du &= 2 dx & v &= \sin x \end{aligned}$$

Therefore , $\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x - 2 \int \sin x dx$

$$\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

$$5. \int e^x \cos x \, dx$$

$$\begin{aligned} u &= \cos x & dv &= e^x \, dx \\ du &= -\sin x \, dx & v &= e^x \end{aligned}$$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$

$$\text{Now to solve } \int e^x \sin x \, dx$$

$$\begin{aligned} u &= \sin x & dv &= e^x \, dx \\ du &= \cos x \, dx & v &= e^x \end{aligned}$$

$$\text{Therefore, } \int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

$$\int e^x \cos x \, dx = \frac{1}{2} [e^x \cos x + e^x \sin x] + c .$$

$$\text{Another solution of } \int e^x \cos x \, dx$$

$$\begin{aligned} u &= e^x & dv &= \cos x \, dx \\ du &= e^x \, dx & v &= \sin x \end{aligned}$$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

$$\text{Now to solve } \int e^x \sin x \, dx$$

$$\begin{aligned} u &= e^x & dv &= \sin x \, dx \\ du &= e^x \, dx & v &= -\cos x \end{aligned}$$

$$\text{Therefore, } \int e^x \cos x \, dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx \right]$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{1}{2} [e^x \sin x + e^x \cos x] + c .$$

$$6. \int \ln |x| \, dx$$

$$\begin{aligned} u &= \ln |x| & dv &= dx \\ du &= \frac{1}{x} \, dx & v &= x \end{aligned}$$

$$\int \ln |x| \, dx = x \ln |x| - \int x \frac{1}{x} \, dx = x \ln |x| - \int dx = x \ln |x| - x + c$$

$$\begin{aligned}
7. \quad & \int \tan^{-1} x \, dx \\
& u = \tan^{-1} x \quad dv = dx \\
& du = \frac{1}{1+x^2} \, dx \quad v = x \\
& \int \tan^{-1} x \, dx = x \tan^{-1} x - \int x \frac{1}{1+x^2} \, dx \\
& \int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + c
\end{aligned}$$

$$\begin{aligned}
8. \quad & \int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx \\
& u = \sec x \quad dv = \sec^2 x \, dx \\
& du = \sec x \tan x \, dx \quad v = \tan x \\
& \int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx \\
& \int \sec^3 x \, dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx \\
& \int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \\
& 2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x| \\
& \int \sec^3 x \, dx = \frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|] + c
\end{aligned}$$

$$\begin{aligned}
9. \quad & \int \ln(1+x^2) \, dx \\
& u = \ln(1+x^2) \quad dv = dx \\
& du = \frac{2x}{1+x^2} \, dx \quad v = x \\
& \int \ln(1+x^2) \, dx = x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} \, dx \\
& \int \ln(1+x^2) \, dx = x \ln(1+x^2) - \int \frac{(2x^2+2)-2}{1+x^2} \, dx \\
& \int \ln(1+x^2) \, dx = x \ln(1+x^2) - \int \frac{2(x^2+1)}{1+x^2} \, dx + 2 \int \frac{1}{1+x^2} \, dx \\
& \int \ln(1+x^2) \, dx = x \ln(1+x^2) - 2x + 2 \tan^{-1} x + c
\end{aligned}$$

$$\begin{aligned}
10. \quad & \int \frac{x^3}{\sqrt{x^2+1}} \, dx = \int x^2 \frac{x}{\sqrt{x^2+1}} \, dx \\
& u = x^2 \quad dv = \frac{x}{\sqrt{x^2+1}} \, dx \\
& du = 2x \, dx \quad v = \sqrt{x^2+1}
\end{aligned}$$

$$\int \frac{x^3}{\sqrt{x^2+1}} dx = x^2 \sqrt{x^2+1} - \int 2x \sqrt{x^2+1} dx$$

$$\int \frac{x^3}{\sqrt{x^2+1}} dx = x^2 \sqrt{x^2+1} - \int (x^2+1)^{\frac{1}{2}} 2x dx$$

$$\int \frac{x^3}{\sqrt{x^2+1}} dx = x^2 \sqrt{x^2+1} - \frac{(x^2+1)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$11. \int x^3 e^{x^2} dx = \int x^2 (x e^{x^2}) dx$$

$$u = x^2 \quad dv = x e^{x^2} dx$$

$$du = 2x dx \quad v = \frac{1}{2} e^{x^2}$$

$$\int x^3 e^{x^2} dx = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} \int 2x e^{x^2} dx$$

$$\int x^3 e^{x^2} dx = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + c$$

Exercises : Solve the following integrals

$$1. \int x \cos 2x dx .$$

$$2. \int x \cosh x dx .$$

$$3. \int \frac{x}{e^x} dx .$$

$$4. \int e^x \sin x dx .$$

$$5. \int \frac{1}{x^2} \ln |x| dx .$$

$$6. \int \sin^{-1} x dx .$$

Notes :

1. $\int x e^x dx = (x - 1)e^x + c .$

$$\int x^2 e^x dx = (x^2 - 2x + 2)e^x + c .$$

$$\int x^3 e^x dx = (x^3 - 3x^2 + 6x - 6)e^x + c .$$

2. $\int x \cos x dx = x \sin x + \cos x + c$

$$\int x^2 \cos x dx = (x^2 - 2) \sin x + 2x \cos x + c$$

$$\int x^3 \cos x dx = (x^3 - 6x) \sin x + (3x^2 - 6) \cos x + c$$

$$\int x^4 \cos x dx = (x^4 - 12x^2 + 24) \sin x + (4x^3 - 24x) \cos x + c$$

3. $\int x \sin x dx = -x \cos x + \sin x + c$

$$\int x^2 \sin x dx = (-x^2 + 2) \cos x + 2x \sin x + c$$

$$\int x^3 \sin x dx = (-x^3 + 6x) \cos x + (3x^2 - 6) \sin x + c$$

$$\int x^4 \sin x dx = (-x^4 + 12x^2 - 24) \cos x + (4x^3 - 24x) \sin x + c$$

INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

FIRST : Integrals of the forms

$$\int \sin ax \cos bx \, dx \quad , \quad \int \sin ax \sin bx \, dx \quad , \quad \int \cos ax \cos bx \, dx$$

Where $a, b \in \mathbb{Z}$.

1. The integral $\int \sin ax \cos bx \, dx$ can be solved using the formula

$$\sin ax \cos bx = \frac{1}{2} [\sin(ax + bx) + \sin(ax - bx)]$$

2. The integral $\int \sin ax \sin bx \, dx$ can be solved using the formula

$$\sin ax \sin bx = \frac{1}{2} [\cos(ax - bx) - \cos(ax + bx)]$$

3. The integral $\int \cos ax \cos bx \, dx$ can be solved using the formula

$$\cos ax \cos bx = \frac{1}{2} [\cos(ax + bx) + \cos(ax - bx)]$$

Examples :

1.
$$\begin{aligned} \int \sin 3x \cos 2x \, dx &= \int \frac{1}{2} [\sin(3x + 2x) + \sin(3x - 2x)] \, dx \\ &= \int \frac{1}{2} [\sin 5x + \sin x] \, dx = \frac{1}{2} \int \sin 5x \, dx + \frac{1}{2} \int \sin x \, dx \\ &= \frac{1}{2} \frac{1}{5} (-\cos 5x) + \frac{1}{2} (-\cos x) + c = -\frac{1}{10} \cos 5x - \frac{1}{2} \cos x + c \end{aligned}$$

2.
$$\begin{aligned} \int \sin x \sin 3x \, dx &= \int \frac{1}{2} [\cos(3x - x) - \cos(3x + x)] \, dx \\ &= \int \frac{1}{2} [\cos 2x - \cos 4x] \, dx = \frac{1}{2} \int \cos 2x \, dx - \frac{1}{2} \int \cos 4x \, dx \\ &= \frac{1}{2} \frac{1}{2} \sin 2x - \frac{1}{2} \frac{1}{4} \sin 4x + c = \frac{1}{4} \sin 2x - \frac{1}{8} \sin 4x + c \end{aligned}$$

3.
$$\begin{aligned} \int \cos 5x \cos 2x \, dx &= \int \frac{1}{2} [\cos(5x + 2x) + \cos(5x - 2x)] \, dx \\ &= \int \frac{1}{2} [\cos 7x + \cos 3x] \, dx = \frac{1}{2} \int \cos 7x \, dx + \frac{1}{2} \int \cos 3x \, dx \\ &= \frac{1}{2} \frac{1}{7} \sin 7x + \frac{1}{2} \frac{1}{3} \sin 3x + c = \frac{1}{14} \sin 7x + \frac{1}{6} \sin 3x + c \end{aligned}$$

SECOND : Integrals of the forms

$$\int \sin^n x \cos^m x dx , \quad \int \sinh^n x \cosh^m x dx , \quad \text{where } n, m \in \mathbb{N}$$

The above two integrals can be solved by substitution if n or m is odd .

1. If n is odd :

The substitution $u = \cos x$ can be used to solve $\int \sin^n x \cos^m x dx$.

The substitution $u = \cosh x$ can be used to solve $\int \sinh^n x \cosh^m x dx$.

2. If m is odd :

The substitution $u = \sin x$ can be used to solve $\int \sin^n x \cos^m x dx$.

The substitution $u = \sinh x$ can be used to solve $\int \sinh^n x \cosh^m x dx$.

Examples :

$$\begin{aligned} 1. \quad & \int \sin^5 x \cos^4 x dx = \int \sin^4 x \cos^4 x \sin x dx \\ & = \int (\sin^2 x)^2 \cos^4 x \sin x dx = \int (1 - \cos^2 x)^2 \cos^4 x \sin x dx \\ & \text{Put } u = \cos x \Rightarrow -du = \sin x dx \\ & \int \sin^5 x \cos^4 x dx = - \int (1 - u^2)^2 u^4 du = - \int (1 - 2u^2 + u^4) u^4 du \\ & = - \int (u^4 - 2u^6 + u^8) du = - \left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} \right] + c \\ & = -\frac{\cos^5 x}{5} + \frac{2 \cos^7 x}{7} - \frac{\cos^9 x}{9} + c \end{aligned}$$

$$\begin{aligned} 2. \quad & \int \sqrt{\sin x} \cos^3 x dx = \int \sqrt{\sin x} \cos^2 x \cos x dx \\ & = \int (\sin x)^{\frac{1}{2}} (1 - \sin^2 x) \cos x dx \\ & \text{Put } u = \sin x \Rightarrow du = \cos x dx \\ & \int \sqrt{\sin x} \cos^3 x dx = \int u^{\frac{1}{2}} (1 - u^2) du = \int \left(u^{\frac{1}{2}} - u^{\frac{5}{2}} \right) du \\ & = \frac{2u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{7}{2}}}{7} + c = \frac{2(\sin x)^{\frac{3}{2}}}{3} - \frac{2(\sin x)^{\frac{7}{2}}}{7} + c \end{aligned}$$

$$3. \int \frac{\sin^3 x}{\cos^2 x} dx = \int \sin^2 x \cos^{-2} x \sin x dx = \int (1 - \cos^2 x) \cos^{-2} x \sin x dx$$

$$\text{Put } u = \cos x \Rightarrow -du = \sin x dx$$

$$\int \frac{\sin^3 x}{\cos^2 x} dx = - \int (1 - u^2) u^{-2} du = - \int (u^{-2} - 1) du$$

$$= -\frac{u^{-1}}{-1} + u + c = \frac{1}{u} + u + c = \sec x + \cos x + c$$

$$4. \int \sinh^3 x \cosh^2 x dx = \int \sinh^2 x \cosh^2 x \sinh x dx$$

$$= \int (\cosh^2 x - 1) \cosh^2 x \sinh x dx$$

$$\text{Put } u = \cosh x \Rightarrow du = \sinh x dx$$

$$\int \sinh^3 x \cosh^2 x dx = \int (u^2 - 1) u^2 du = \int (u^4 - u^2) du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + c = \frac{\cosh^5 x}{5} - \frac{\cosh^3 x}{3} + c$$

$$5. \int \sin^7 x \cos^3 x dx = \int \sin^6 x \cos^2 x \cos x dx$$

$$= \int \sin^6 x (1 - \sin^2 x) \cos x dx$$

$$\text{Put } u = \sin x \Rightarrow du = \cos x dx$$

$$\int \sin^7 x \cos^3 x dx = \int u^7 (1 - u^2) du = \int (u^7 - u^9) du$$

$$= \frac{u^8}{8} - \frac{u^{10}}{10} + c = \frac{\sin^8 x}{8} - \frac{\sin^{10} x}{10} + c$$

Special cases :

$$1. \int \sin^2 x dx = \int \frac{1}{2} [1 - \cos 2x] dx = \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right] + c$$

$$2. \int \cos^2 x dx = \int \frac{1}{2} [1 + \cos 2x] dx = \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right] + c$$

Exercises : Solve the following integrals

$$1. \int \sin^3 x dx$$

$$2. \int \sin^2 x \cos^5 x dx$$

THIRD : Integrals of the forms

$$\int \sec^n x \tan^m x dx , \quad \int \csc^n x \cot^m x dx ,$$

$$\int \operatorname{sech}^n x \tanh^m x dx , \quad \int \operatorname{csch}^n x \coth^m x dx$$

The above four integrals can be solved by substitution if n is even or m is odd .

1. If n is even :

The substitution $u = \tan x$ can be used to solve $\int \sec^n x \tan^m x dx$.

The substitutions $u = \cot x$, $u = \tanh x$ and $u = \coth x$ can be used to solve the other three integrals respectively.

2. If m is odd :

The substitution $u = \sec x$ can be used to solve $\int \sec^n x \tan^m x dx$.

The substitutions $u = \csc x$, $u = \operatorname{sech} x$ and $u = \operatorname{csch} x$ can be used to solve the other three integrals respectively.

Examples :

1. $\int \csc^4 x \cot^4 x dx$

$$= \int \csc^2 x \cot^4 x \csc^2 x dx = \int (1 + \cot^2 x) \cot^4 x \csc^2 x dx$$

Put $u = \cot x \Rightarrow -du = \csc^2 x dx$

$$\int \csc^4 x \cot^4 x dx = - \int (1 + u^2) u^4 du = - \int (u^4 + u^6) du$$

$$= -\frac{u^5}{5} - \frac{u^7}{7} + c = -\frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + c$$

2. $\int \tan^3 x \sec^3 x dx$

$$= \int \tan^2 x \sec^2 x \sec x \tan x dx = \int (\sec^2 x - 1) \sec^2 x \sec x \tan x dx$$

Put $u = \sec x \Rightarrow du = \sec x \tan x dx$

$$\int \tan^3 x \sec^3 x dx = \int (u^2 - 1) u^2 du = \int (u^4 - u^2) du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + c = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c$$

3. $\int \tanh^3 x \operatorname{sech} x \, dx$
- $$= \int \tanh^2 x \operatorname{sech} x \tanh x \, dx = \int (1 - \operatorname{sech}^2 x) \operatorname{sech} x \tanh x \, dx$$
- Put $u = \operatorname{sech} x \Rightarrow -du = \operatorname{sech} x \tanh x \, dx$
- $$\int \tanh^3 x \operatorname{sech} x \, dx = - \int (1 - u^2) \, du$$
- $$= -u + \frac{u^3}{3} + c = -\operatorname{sech} x + \frac{\operatorname{sech}^3 x}{3} + c$$
4. $\int \frac{\sec^4 x}{\sqrt{\tan x}} \, dx$
- $$\int \sec^2 x (\tan x)^{-\frac{1}{2}} \sec^2 x \, dx = \int (1 + \tan^2 x) (\tan x)^{-\frac{1}{2}} \sec^2 x \, dx$$
- Put $u = \tan x \Rightarrow du = \sec^2 x \, dx$
- $$\int \frac{\sec^4 x}{\sqrt{\tan x}} \, dx = \int (1 + u^2) u^{-\frac{1}{2}} \, du = \int \left(u^{-\frac{1}{2}} + u^{\frac{3}{2}} \right) \, du$$
- $$= 2u^{\frac{1}{2}} + \frac{2u^{\frac{5}{2}}}{5} + c = 2(\tan x)^{\frac{1}{2}} + \frac{2(\tan x)^{\frac{5}{2}}}{5} + c$$
5. $\int \tan^4 x \sec^2 x \, dx = \int (\tan x)^4 \sec^2 x \, dx = \frac{\tan^5 x}{5} + c$

TRIGONOMETRIC SUBSTITUTIONS

If the integrand contains a term of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ where $a > 0$, then trigonometric substitutions can be used to solve the integral.

1. An integral involving $\sqrt{a^2 - x^2}$: use the substitution $x = a \sin \theta$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ to solve the integral.
2. An integral involving $\sqrt{a^2 + x^2}$: use the substitution $x = a \tan \theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to solve the integral.
3. An integral involving $\sqrt{x^2 - a^2}$: use the substitution $x = a \sec \theta$ where $\theta \in \left[0, \frac{\pi}{2}\right)$ to solve the integral.

Examples :

1. To solve the integral $\int \frac{\sqrt{x^2 - 9}}{x} dx$ we use the substitution :
 (a) $x = 3 \tan \theta$ (b) $x = 3 \sin \theta$ (c) $x = 3 \sec \theta$ (d) None of these

Answer : We use the substitution $x = 3 \sec \theta$.

2. To solve the integral $\int \sqrt{1 + 4x^2} dx$ we use the substitution :
 (a) $2x = \cos \theta$ (b) $x = \frac{\tan \theta}{2}$ (c) $2x = \sin \theta$ (d) None of these

Answer : $\sqrt{1 + 4x^2} = \sqrt{(1)^2 + (2x)^2}$

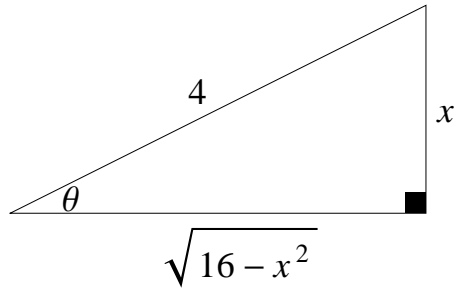
So we use the substitution $2x = \tan \theta \Rightarrow x = \frac{\tan \theta}{2}$

3. $\int \frac{1}{x^2 \sqrt{16 - x^2}} dx = \int \frac{1}{x^2 \sqrt{(4)^2 - x^2}} dx$

Put $x = 4 \sin \theta \Rightarrow \sin \theta = \frac{x}{4}$

$dx = 4 \cos \theta d\theta$

$$\begin{aligned} \int \frac{1}{x^2 \sqrt{16 - x^2}} dx &= \int \frac{4 \cos \theta}{(4 \sin \theta)^2 \sqrt{16 - (4 \sin \theta)^2}} d\theta \\ &= \int \frac{4 \cos \theta}{16 \sin^2 \theta \sqrt{16 - 16 \sin^2 \theta}} d\theta = \int \frac{4 \cos \theta}{16 \sin^2 \theta \sqrt{16(1 - \sin^2 \theta)}} d\theta \\ &= \int \frac{4 \cos \theta}{16 \sin^2 \theta 4 \cos \theta} d\theta = \frac{1}{16} \int \frac{1}{\sin^2 \theta} d\theta = \frac{1}{16} \int \csc^2 \theta d\theta \\ &= -\frac{1}{16} \cot \theta + c \end{aligned}$$



$$\int \frac{1}{x^2 \sqrt{16-x^2}} dx = -\frac{1}{16} \frac{\sqrt{16-x^2}}{x} + c$$

4. $\int \frac{\sqrt{x^2-4}}{x^2} dx$

Put $x = 2 \sec \theta \Rightarrow \sec \theta = \frac{x}{2}$

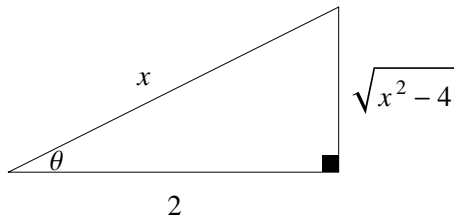
$$dx = 2 \sec \theta \tan \theta d\theta$$

$$\int \frac{\sqrt{x^2-4}}{x^2} dx = \int \frac{\sqrt{4 \sec^2 \theta - 4} \cdot 2 \sec \theta \tan \theta}{4 \sec^2 \theta} d\theta$$

$$= \int \frac{(2 \tan \theta)(2 \sec \theta \tan \theta)}{4 \sec^2 \theta} d\theta = \int \frac{\tan^2 \theta}{\sec \theta} d\theta$$

$$= \int \frac{(\sec^2 \theta - 1)}{\sec \theta} d\theta = \int \frac{\sec^2 \theta}{\sec \theta} d\theta - \int \frac{1}{\sec \theta} d\theta$$

$$= \int \sec \theta d\theta - \int \cos \theta d\theta = \ln |\sec \theta + \tan \theta| - \sin \theta + c$$



$$\int \frac{\sqrt{x^2-4}}{x^2} dx = \ln \left| \frac{x}{2} + \frac{\sqrt{x^2-4}}{2} \right| - \frac{\sqrt{x^2-4}}{x} + c$$

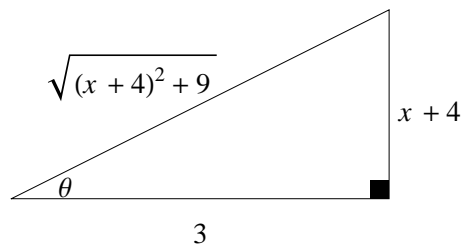
5. $\int \frac{1}{(x^2+8x+25)^{\frac{3}{2}}} dx$

$$= \int \frac{1}{[(x^2+8x+16)+9]^{\frac{3}{2}}} dx = \int \frac{1}{[(x+4)^2+3^2]^{\frac{3}{2}}} dx$$

Put $x+4 = 3 \tan \theta \Rightarrow \tan \theta = \frac{x+4}{3}$

$$dx = 3 \sec^2 \theta \, d\theta$$

$$\begin{aligned} \int \frac{1}{(x^2 + 8x + 25)^{\frac{3}{2}}} dx &= \int \frac{3 \sec^2 \theta}{(9 \tan^2 \theta + 9)^{\frac{3}{2}}} d\theta \\ &= \int \frac{3 \sec^2 \theta}{(9 \sec^2 \theta)^{\frac{3}{2}}} d\theta = \int \frac{3 \sec^2 \theta}{27 \sec^3 \theta} d\theta \\ &= \frac{1}{9} \int \frac{1}{\sec \theta} d\theta = \frac{1}{9} \int \cos \theta \, d\theta = \frac{1}{9} \sin \theta + c \end{aligned}$$



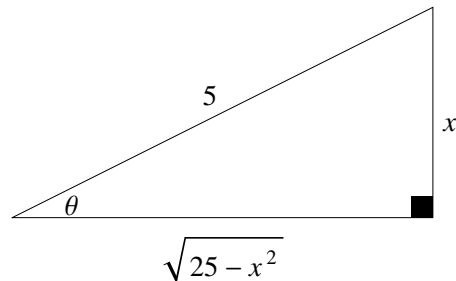
$$\int \frac{1}{(x^2 + 8x + 25)^{\frac{3}{2}}} dx = \frac{1}{9} \frac{x + 4}{\sqrt{(x + 4)^2 + 9}} + c = \frac{1}{9} \frac{x + 4}{\sqrt{x^2 + 8x + 25}} + c$$

6. $\int \frac{1}{(25 - x^2)^{\frac{3}{2}}} dx$

Put $x = 5 \sin \theta \Rightarrow \sin \theta = \frac{x}{5}$

$$dx = 5 \cos \theta \, d\theta$$

$$\begin{aligned} \int \frac{1}{(25 - x^2)^{\frac{3}{2}}} dx &= \int \frac{5 \cos \theta}{(25 - 25 \sin^2 \theta)^{\frac{3}{2}}} d\theta \\ &= \int \frac{5 \cos \theta}{(25 \cos^2 \theta)^{\frac{3}{2}}} d\theta = \int \frac{5 \cos \theta}{125 \cos^3 \theta} d\theta \\ &= \frac{1}{25} \int \frac{1}{\cos^2 \theta} d\theta = \frac{1}{25} \int \sec^2 \theta \, d\theta = \frac{1}{25} \tan \theta + c \end{aligned}$$



$$\int \frac{1}{(25 - x^2)^{\frac{3}{2}}} dx = \frac{1}{25} \frac{x}{\sqrt{25 - x^2}} + c$$

$$7. \int \frac{x}{\sqrt{x^2 - 16}} dx = \frac{1}{2} \int (x^2 - 16)^{-\frac{1}{2}} 2x dx = \sqrt{x^2 - 16} + c$$

Notes :

$$1. \int \frac{1}{\sqrt{9 - x^2}} dx$$

$$\text{Put } x = 3 \sin \theta \Rightarrow \sin \theta = \frac{x}{3}$$

$$dx = 3 \cos \theta d\theta$$

$$\int \frac{1}{\sqrt{9 - x^2}} dx = \int \frac{3 \cos \theta}{\sqrt{9 - 9 \sin^2 \theta}} d\theta$$

$$= \int \frac{3 \cos \theta}{3 \cos \theta} d\theta = \int d\theta = \theta + c = \sin^{-1} \left(\frac{x}{3} \right) + c$$

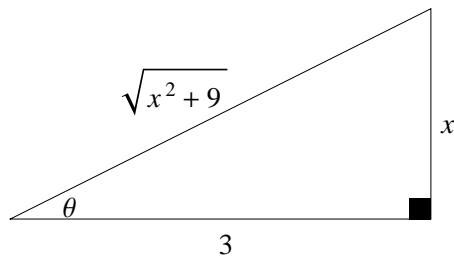
$$2. \int \frac{1}{\sqrt{9 + x^2}} dx$$

$$\text{Put } x = 3 \tan \theta \Rightarrow \tan \theta = \frac{x}{3}$$

$$dx = 3 \sec^2 \theta d\theta$$

$$\int \frac{1}{\sqrt{9 + x^2}} dx = \int \frac{3 \sec^2 \theta}{\sqrt{9 + 9 \tan^2 \theta}} d\theta$$

$$= \int \frac{3 \sec^2 \theta}{3 \sec \theta} d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + c$$



$$\int \frac{1}{\sqrt{9 + x^2}} dx = \ln \left| \frac{\sqrt{x^2 + 9}}{3} + \frac{x}{3} \right| + c$$

Exercises : Solve the following integrals

1. $\int \frac{x^2}{\sqrt{4-x^2}} dx$

Hint : use $x = 2 \sin \theta$

2. $\int x^3 \sqrt{x^2-4} dx$

Hint : use $x = 2 \sec \theta$

3. $\int \sqrt{x^2+2x+2} dx$

Hint : use $x+1 = \tan \theta$

4. $\int \frac{1}{\sqrt{x^2+2x+5}} dx$

Hint : use $x+1 = 2 \tan \theta$

5. $\int \frac{x^3}{\sqrt{9x^2+49}} dx$

Hint : use $3x = 7 \tan \theta$

INTEGRATION OF RATIONAL FUNCTIONS (Method of Partial fractions)

Method of partial fractions is used to solve integrals of the form $\int \frac{P(x)}{Q(x)} dx$ where $P(x)$, $Q(x)$ are polynomials and *degree* $P(x) < \text{degree } Q(x)$.
If *degree* $P(x) \geq \text{degree } Q(x)$ use long division of polynomials .

Definition (linear factor) :

A linear factor is a polynomial of degree 1.
It has the form $ax + b$ where $a, b \in \mathbb{R}$ and $a \neq 0$.

Examples :

x , $3x$, $2x - 7$ are examples of linear factors .

Definition (irreducible quadratic) :

An irreducible quadratic is a polynomial of degree 2.
It has the form $ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$, $a \neq 0$ and $b^2 - 4ac < 0$.

Examples :

1. $x^2 + 9$ and $x^2 + x + 1$ are examples of irreducible quadratics.
2. $x^2 = x \cdot x$ and $x^2 - 1 = (x - 1)(x + 1)$ are reducible quadratics .

How to write $\frac{P(x)}{Q(x)}$ as partial fractions decomposition ?

Write $Q(x)$ as a product of linear factors and irreducible quadratics (if possible).

If $Q(x) = (a_1x + a_2)^m (b_1x^2 + b_2x + b_3)^n$ where $m, n \in \mathbb{N}$ then

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + a_2} + \frac{A_2}{(a_1x + a_2)^2} + \dots + \frac{A_m}{(a_1x + a_2)^m} + \frac{B_1x + C_1}{b_1x^2 + b_2x + b_3} + \frac{B_2x + C_2}{(b_1x^2 + b_2x + b_3)^2} + \dots + \frac{B_nx + C_n}{(b_1x^2 + b_2x + b_3)^n}$$

Where $A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_n, C_1, C_2, \dots, C_n \in \mathbb{R}$.

Examples : Write the partial fractions decomposition of the following

1. $\frac{2x + 6}{x^2 - 2x - 3} = \frac{2x + 6}{(x - 3)(x + 1)} = \frac{A_1}{x - 3} + \frac{A_2}{x + 1}$
2. $\frac{x + 5}{x^2 + 4x + 4} = \frac{x + 5}{(x + 2)^2} = \frac{A_1}{x + 2} + \frac{A_2}{(x + 2)^2}$
3. $\frac{x^2 + 1}{x^4 + 4x^2} = \frac{x^2 + 1}{x^2(x^2 + 4)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1x + C_1}{x^2 + 4}$
4. $\frac{2x + 7}{(x + 1)(x^2 + 9)^2} = \frac{A_1}{x + 1} + \frac{B_1x + C_1}{x^2 + 9} + \frac{B_2x + C_2}{(x^2 + 9)^2}$
5. $\frac{x}{(x - 1)(x^2 - 1)} = \frac{x}{(x + 1)(x - 1)^2} = \frac{A_1}{x + 1} + \frac{A_2}{x - 1} + \frac{A_3}{(x - 1)^2}$

$$\begin{aligned}
6. \quad \frac{x^4 + 2x^3 + 1}{x^4 + x^3 + x^2} &= \frac{(x^4 + x^3 + x^2) + (x^3 - x^2 + 1)}{x^4 + x^3 + x^2} = 1 + \frac{x^3 - x^2 + 1}{x^4 + x^3 + x^2} \\
&= 1 + \frac{x^3 - x^2 + 1}{x^2(x^2 + x + 1)} = 1 + \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1x + C_1}{x^2 + x + 1}
\end{aligned}$$

Examples :

$$1. \int \frac{x^2 + 12x + 3}{x^3 - 4x} dx$$

$$\frac{x^2 + 12x + 3}{x^3 - 4x} = \frac{x^2 + 12x + 3}{x(x-2)(x+2)} = \frac{A_1}{x} + \frac{A_2}{x-2} + \frac{A_3}{x+2}$$

$$\frac{x^2 + 12x + 3}{x^3 - 4x} = \frac{A_1(x-2)(x+2)}{x(x-2)(x+2)} + \frac{A_2 x(x+2)}{x(x-2)(x+2)} + \frac{A_3 x(x-2)}{x(x-2)(x+2)}$$

$$x^2 + 12x + 3 = A_1(x-2)(x+2) + A_2 x(x+2) + A_3 x(x-2) \rightarrow (*)$$

Put $x = 0$ in equation (*) : $(0)^2 + 12(0) + 3 = A_1(0-2)(0+2) + 0 + 0$

$$3 = -4A_1 \Rightarrow A_1 = -\frac{3}{4}$$

Put $x = 2$ in equation (*) : $(2)^2 + 12(2) + 3 = 0 + A_2 2(2+2) + 0$

$$31 = 8A_2 \Rightarrow A_2 = \frac{31}{8}$$

Put $x = -2$ in equation (*) : $(-2)^2 + 12(-2) + 3 = 0 + 0 + A_3 (-2)(-2-2)$

$$-17 = 8A_3 \Rightarrow A_3 = -\frac{17}{8}$$

$$\frac{x^2 + 12x + 3}{x^3 - 4x} = \frac{-\frac{3}{4}}{x} + \frac{\frac{31}{8}}{x-2} + \frac{-\frac{17}{8}}{x+2}$$

$$\frac{x^2 + 12x + 3}{x^3 - 4x} = -\frac{3}{4} \frac{1}{x} + \frac{31}{8} \frac{1}{x-2} - \frac{17}{8} \frac{1}{x+2}$$

$$\int \frac{x^2 + 12x + 3}{x^3 - 4x} dx = \int \left[-\frac{3}{4} \frac{1}{x} + \frac{31}{8} \frac{1}{x-2} - \frac{17}{8} \frac{1}{x+2} \right] dx$$

$$\int \frac{x^2 + 12x + 3}{x^3 - 4x} dx = -\frac{3}{4} \int \frac{1}{x} dx + \frac{31}{8} \int \frac{1}{x-2} dx - \frac{17}{8} \int \frac{1}{x+2} dx$$

$$\int \frac{x^2 + 12x + 3}{x^3 - 4x} dx = -\frac{3}{4} \ln|x| + \frac{31}{8} \ln|x-2| - \frac{17}{8} \ln|x+2| + c$$

$$2. \int \frac{4}{x^4 - x^3} dx$$

$$\frac{4}{x^4 - x^3} = \frac{4}{x^3(x-1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{A_4}{x-1}$$

$$\frac{4}{x^4 - x^3} = \frac{A_1 x^2(x-1)}{x^3(x-1)} + \frac{A_2 x(x-1)}{x^3(x-1)} + \frac{A_3(x-1)}{x^3(x-1)} + \frac{A_4 x^3}{x^3(x-1)}$$

$$4 = A_1 x^2(x-1) + A_2 x(x-1) + A_3(x-1) + A_4 x^3$$

$$4 = A_1 x^3 - A_1 x^2 + A_2 x^2 - A_2 x + A_3 x - A_3 + A_4 x^3$$

$$4 = (A_1 + A_4)x^3 + (A_2 - A_1)x^2 + (A_3 - A_2)x - A_3$$

By comparing the coefficients of both sides :

$$A_1 + A_4 = 0 \longrightarrow (1)$$

$$A_2 - A_1 = 0 \longrightarrow (2)$$

$$A_3 - A_2 = 0 \longrightarrow (3)$$

$$-A_3 = 4 \longrightarrow (4)$$

From equation (4) : $A_3 = -4$

From equation (3) : $A_2 = A_3 = -4$

From equation (2) : $A_1 = A_2 = -4$

From equation (1) : $A_4 = -A_1 = 4$

$$\frac{4}{x^4 - x^3} = \frac{-4}{x} + \frac{-4}{x^2} + \frac{-4}{x^3} + \frac{4}{x-1}$$

$$\int \frac{4}{x^4 - x^3} dx = -4 \int \frac{1}{x} dx - 4 \int x^{-2} dx - 4 \int x^{-3} dx + 4 \int \frac{1}{x-1} dx$$

$$\int \frac{4}{x^4 - x^3} dx = -4 \ln|x| - 4 \frac{x^{-1}}{-1} - 4 \frac{x^{-2}}{-2} + 4 \ln|x-1| + c$$

$$\int \frac{4}{x^4 - x^3} dx = -4 \ln|x| + \frac{4}{x} + \frac{2}{x^2} + 4 \ln|x-1| + c$$

$$3. \int \frac{8}{(x^2+1)(x^2+9)} dx$$

$$\frac{8}{(x^2+1)(x^2+9)} = \frac{B_1x + C_1}{x^2+1} + \frac{B_2x + C_2}{x^2+9}$$

$$\frac{8}{(x^2+1)(x^2+9)} = \frac{(B_1x + C_1)(x^2+9)}{(x^2+1)(x^2+9)} + \frac{(B_2x + C_2)(x^2+1)}{(x^2+1)(x^2+9)}$$

$$8 = (B_1x + C_1)(x^2+9) + (B_2x + C_2)(x^2+1)$$

$$8 = B_1x^3 + 9B_1x + C_1x^2 + 9C_1 + B_2x^3 + B_2x + C_2x^2 + C_2$$

$$8 = (B_1 + B_2)x^3 + (C_1 + C_2)x^2 + (9B_1 + B_2)x + (9C_1 + C_2)$$

By comparing the coefficients of both sides :

$$B_1 + B_2 = 0 \longrightarrow (1)$$

$$C_1 + C_2 = 0 \longrightarrow (2)$$

$$9B_1 + B_2 = 0 \longrightarrow (3)$$

$$9C_1 + C_2 = 8 \longrightarrow (4)$$

Equation (3) - Equation (1) : $8B_1 = 0 \Rightarrow B_1 = 0$

From equation (1) : $B_2 = -B_1 = 0$

Equation (4) - Equation (2) : $8C_1 = 8 \Rightarrow C_1 = 1$

From equation (2) : $C_2 = -C_1 = -1$

$$\frac{8}{(x^2 + 1)(x^2 + 9)} = \frac{1}{x^2 + 1} + \frac{-1}{x^2 + 9}$$

$$\int \frac{8}{(x^2 + 1)(x^2 + 9)} dx = \int \frac{1}{x^2 + 1} dx - \int \frac{1}{x^2 + 9} dx$$

$$\int \frac{8}{(x^2 + 1)(x^2 + 9)} dx = \tan^{-1} x - \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) + c$$

4. $\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx$

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{B_1x + C_1}{x^2 + 2} + \frac{B_2x + C_2}{(x^2 + 2)^2}$$

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{(B_1x + C_1)(x^2 + 2)}{(x^2 + 2)^2} + \frac{B_2x + C_2}{(x^2 + 2)^2}$$

$$8x^3 + 13x = (B_1x + C_1)(x^2 + 2) + B_2x + C_2$$

$$8x^3 + 13x = B_1x^3 + 2B_1x + C_1x^2 + 2C_1 + B_2x + C_2$$

$$8x^3 + 13x = B_1x^3 + C_1x^2 + (2B_1 + B_2)x + (2C_1 + C_2)$$

By comparing the coefficients of both sides :

$$B_1 = 8$$

$$C_1 = 0$$

$$2B_1 + B_2 = 13 \Rightarrow B_2 = 13 - 2(8) = 13 - 16 = -3$$

$$2C_1 + C_2 = 0 \Rightarrow C_2 = 0 - 2(0) = 0$$

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{8x}{x^2 + 2} + \frac{-3x}{(x^2 + 2)^2}$$

$$\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx = 4 \int \frac{2x}{x^2 + 2} dx - \frac{3}{2} \int \frac{2x}{(x^2 + 2)^2} dx$$

$$\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx = 4 \ln(x^2 + 2) - \frac{3}{2} \frac{(x^2 + 2)^{-1}}{-1} + c$$

$$\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx = 4 \ln(x^2 + 2) + \frac{3}{2} \frac{1}{x^2 + 2} + c$$

5. $\int \frac{x^3 + 1}{x^3 + 4x} dx$

$$\frac{x^3 + 1}{x^3 + 4x} = \frac{(x^3 + 4x) + (1 - 4x)}{x^3 + 4x} = 1 + \frac{1 - 4x}{x^3 + 4x} = 1 + \frac{1 - 4x}{x(x^2 + 4)}$$

$$\frac{1-4x}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+c}{x^2+4}$$

$$\frac{1-4x}{x(x^2+4)} = \frac{A(x^2+4)}{x(x^2+4)} + \frac{(Bx+c)x}{x(x^2+4)}$$

$$1-4x = A(x^2+4) + (Bx+C)x = Ax^2 + 4A + Bx^2 + Cx$$

$$1-4x = (A+B)x^2 + Cx + 4A$$

By comparing the coefficients of both sides :

$$4A = 1 \Rightarrow A = \frac{1}{4}$$

$$C = -4$$

$$A + B = 0 \Rightarrow B = -A = -\frac{1}{4}$$

$$\frac{x^3+1}{x^3+4x} = 1 + \frac{\frac{1}{4}}{x} + \frac{-\frac{1}{4}x-4}{x^2+4}$$

$$\frac{x^3+1}{x^3+4x} = 1 + \frac{1}{4} \frac{1}{x} - \frac{1}{4} \frac{x}{x^2+4} - 4 \frac{1}{x^2+4}$$

$$\int \frac{x^3+1}{x^3+4x} dx = \int 1 dx + \frac{1}{4} \int \frac{1}{x} dx - \frac{1}{8} \int \frac{2x}{x^2+4} dx - 4 \int \frac{1}{x^2+4} dx$$

$$\int \frac{x^3+1}{x^3+4x} dx = x + \frac{1}{4} \ln|x| - \frac{1}{8} \ln(x^2+4) - 4 \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$$

6. $\int \frac{3 \cos x}{\sin^2 x + \sin x - 2} dx$

Put $u = \sin x \Rightarrow du = \cos x dx$

$$\int \frac{3 \cos x}{\sin^2 x + \sin x - 2} dx = \int \frac{3}{u^2 + u - 2} du$$

$$\frac{3}{u^2 + u - 2} = \frac{3}{(u-1)(u+2)} = \frac{1}{u-1} + \frac{-1}{u+2}$$

$$\int \frac{3}{u^2 + u - 2} du = \int \frac{1}{u-1} du - \int \frac{1}{u+2} du$$

$$\int \frac{3}{u^2 + u - 2} du = \ln|u-1| - \ln|u+2| + c$$

$$\int \frac{3 \cos x}{\sin^2 x + \sin x - 2} dx = \ln|\sin x - 1| - \ln|\sin x + 2| + c$$

Exercises : Solve the following integrals

1. $\int \frac{1}{x^2 - 3x + 2} dx$

2. $\int \frac{3}{(x^2 + 1)(x^2 + 4)} dx$

3. $\int \frac{e^x}{(e^x - 1)(e^x + 4)} dx$

HALF-ANGLE SUBSTITUTION

It is used to solve integrals of rational functions involving $\sin x$ or $\cos x$, by putting $u = \tan\left(\frac{x}{2}\right)$, in this case $dx = \frac{2}{1+u^2} du$, $\sin x = \frac{2u}{1+u^2}$ and $\cos x = \frac{1-u^2}{1+u^2}$.

Examples :

$$1. \int \frac{1}{2 + \cos x} dx$$

$$\text{Put } u = \tan\left(\frac{x}{2}\right)$$

$$dx = \frac{2}{1+u^2} du \quad \text{and} \quad \cos x = \frac{1-u^2}{1+u^2}$$

$$\int \frac{1}{2 + \cos x} dx = \int \frac{1}{2 + \left(\frac{1-u^2}{1+u^2}\right)} \frac{2}{1+u^2} du$$

$$= \int \frac{1}{\left(\frac{2(1+u^2)+(1-u^2)}{1+u^2}\right)} \frac{2}{1+u^2} du$$

$$= \int \frac{1+u^2}{2+2u^2+1-u^2} \frac{2}{1+u^2} du = \int \frac{2}{3+u^2} du$$

$$= 2 \int \frac{1}{(\sqrt{3})^2 + (u)^2} du = 2 \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + c$$

$$\int \frac{1}{2 + \cos x} dx = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right) + c$$

$$2. \int \frac{1}{3 \sin x + 4 \cos x} dx$$

$$\text{Put } u = \tan\left(\frac{x}{2}\right)$$

$$dx = \frac{2}{1+u^2} du, \quad \cos x = \frac{1-u^2}{1+u^2} \quad \text{and} \quad \sin x = \frac{2u}{1+u^2}$$

$$\int \frac{1}{3 \sin x + 4 \cos x} dx = \int \frac{1}{3\left(\frac{2u}{1+u^2}\right) + 4\left(\frac{1-u^2}{1+u^2}\right)} \frac{2}{1+u^2} du$$

$$= \int \frac{1}{\frac{3(2u)+4(1-u^2)}{1+u^2}} \frac{2}{1+u^2} du = \int \frac{1+u^2}{6u+4-4u^2} \frac{2}{1+u^2} du$$

$$\int \frac{2}{-2(2u^2 - 3u - 2)} du = - \int \frac{1}{(2u+1)(u-2)} du$$

$$\frac{1}{(2u+1)(u-2)} = \frac{A_1}{u-2} + \frac{A_2}{2u+1}$$

$$1 = A_1(2u+1) + A_2(u-2)$$

Put $u = 2$ then $1 = 5A_1 \Rightarrow A_1 = \frac{1}{5}$

Put $u = -\frac{1}{2}$ then $1 = -\frac{5}{2}A_2 \Rightarrow A_2 = -\frac{2}{5}$

$$\frac{1}{(2u+1)(u-2)} = \frac{\frac{1}{5}}{u-2} + \frac{-\frac{2}{5}}{2u+1} = \frac{1}{5} \frac{1}{u-2} - \frac{1}{5} \frac{2}{2u+1}$$

$$-\int \frac{1}{(2u+1)(u-2)} du = -\frac{1}{5} \int \frac{1}{u-2} du + \frac{1}{5} \int \frac{2}{2u+1} du$$

$$= -\frac{1}{5} \ln|u-2| + \frac{1}{5} \ln|2u+1| + c$$

$$\int \frac{1}{3 \sin x + 4 \cos x} dx = -\frac{1}{5} \ln \left| \tan \left(\frac{x}{2} \right) - 2 \right| + \frac{1}{5} \ln \left| 2 \tan \left(\frac{x}{2} \right) + 1 \right| + c$$

3. $\int \frac{1}{1 - \sin x} dx$

$$= \int \frac{1}{1 - \sin x} \frac{1 + \sin x}{1 + \sin x} dx = \int \frac{1 + \sin x}{1 - \sin^2 x} dx$$

$$= \int \frac{1 + \sin x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \right) dx$$

$$= \int \sec^2 x dx + \int \sec x \tan x dx = \tan x + \sec x + c$$

4. $\int \frac{\sin x}{\sqrt{5 - 2 \cos x + \cos^2 x}} dx$

Put $u = \cos x \Rightarrow -du = \sin x$

$$\int \frac{\sin x}{\sqrt{5 - 2 \cos x + \cos^2 x}} dx = \int \frac{-1}{\sqrt{5 - 2u + u^2}} du$$

$$= -\int \frac{1}{\sqrt{(u^2 - 2u + 1) + 4}} du = -\int \frac{1}{\sqrt{(u-1)^2 + (2)^2}} du$$

$$= -\sinh^{-1} \left(\frac{u-1}{2} \right) + c$$

$$\int \frac{\sin x}{\sqrt{5 - 2 \cos x + \cos^2 x}} dx = -\sinh^{-1} \left(\frac{\cos x - 1}{2} \right) + c$$

Exercises : Solve the following integrals

1. $\int \frac{1}{5 + 3 \cos x} dx$

2. $\int \frac{1}{\cos x + \sin x} dx$

3. $\int \frac{1}{\sin x - \cos x - 1} dx$

MISCELLANEOUS SUBSTITUTIONS

1. Integrals involving fraction powers of x

Examples :

1. $\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx$

Put $u = x^{\frac{1}{6}} \Rightarrow x = u^6 \Rightarrow dx = 6u^5 du$

Note that 6 is the least common multiple of 2 and 3

$$\begin{aligned} \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx &= \int \frac{6u^5}{(u^6)^{\frac{1}{2}} + (u^6)^{\frac{1}{3}}} du = \int \frac{6u^5}{u^3 + u^2} du \\ &= \int \frac{6u^5}{u^2(u+1)} du = \int \frac{6u^3}{u+1} du \end{aligned}$$

Use long division of polynomials

$$\begin{aligned} \int \frac{6u^3}{u+1} du &= \int \left(6u^2 - 6u + 6 - \frac{6}{u+1} \right) du \\ &= 2u^3 - 3u^2 + 6u - 6 \ln |u+1| + c \end{aligned}$$

$$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx = 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6 \ln |x^{\frac{1}{6}} + 1| + c$$

2. $\int \frac{x^{\frac{1}{6}}}{x^{\frac{1}{3}} + 1} dx$

Put $u = x^{\frac{1}{6}} \Rightarrow x = u^6 \Rightarrow dx = 6u^5 du$

Note that 6 is the least common multiple of 3 and 6

$$\int \frac{x^{\frac{1}{6}}}{x^{\frac{1}{3}} + 1} dx = \int \frac{u \cdot 6u^5}{u^2 + 1} du = \int \frac{6u^6}{u^2 + 1} du$$

Use long division of polynomials

$$\begin{aligned} \int \frac{6u^6}{u^2+1} du &= \int \left(6u^4 - 6u^2 + 6 - \frac{6}{u^2+1} \right) du \\ &= \frac{6u^5}{5} - 2u^3 + 6u - 6 \tan^{-1} u + c \end{aligned}$$

$$\int \frac{x^{\frac{1}{6}}}{x^{\frac{1}{3}} + 1} dx = \frac{6u^{\frac{5}{6}}}{5} - 2x^{\frac{1}{2}} + 6x^{\frac{1}{6}} - 6 \tan^{-1} \left(x^{\frac{1}{6}} \right) + c$$

2. Integrals involving a square root of a linear factor

Examples :

$$1. \int \frac{1}{(x+1)\sqrt{x-2}} dx$$

$$\text{Put } u = \sqrt{x-2} \Rightarrow x = u^2 + 2 \Rightarrow dx = 2u du$$

$$\int \frac{1}{(x+1)\sqrt{x-2}} dx = \int \frac{2u}{(u^2+3)u} du = \int \frac{2}{u^2+3} du$$

$$= 2 \int \frac{1}{(u)^2 + (\sqrt{3})^2} du = 2 \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{u}{\sqrt{3}} \right) + c$$

$$\int \frac{1}{(x+1)\sqrt{x-2}} dx = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{\sqrt{x-2}}{\sqrt{3}} \right) + c$$

$$2. \int \frac{1}{\sqrt{1+\sqrt{x}}} dx$$

$$\text{Put } u = \sqrt{1+\sqrt{x}} \Rightarrow \sqrt{x} = u^2 - 1 \Rightarrow x = (u^2 - 1)^2 \Rightarrow dx = 4u(u^2 - 1) du$$

$$\int \frac{1}{\sqrt{1+\sqrt{x}}} dx = \int \frac{4u(u^2 - 1)}{u} du = 4 \int (u^2 - 1) du = 4 \left[\frac{u^3}{3} - u \right] + c$$

$$\int \frac{1}{\sqrt{1+\sqrt{x}}} dx = 4 \left[\frac{(\sqrt{1+\sqrt{x}})^3}{3} - \sqrt{1+\sqrt{x}} \right] + c$$

$$3. \int \frac{1-\sqrt{x}}{1+\sqrt{x}} dx$$

$$\text{Put } u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u du$$

$$\int \frac{1-\sqrt{x}}{1+\sqrt{x}} dx = \int \frac{(1-u)2u}{1+u} du = \int \frac{-2u^2+2u}{u+1} du$$

Use long division of polynomials

$$\int \frac{-2u^2+2u}{u+1} du = \int \left(-2u+4 - \frac{4}{u+1} \right) du = -u^2+4u-4 \ln|u+1|+c$$

$$\int \frac{1-\sqrt{x}}{1+\sqrt{x}} dx = -x+4\sqrt{x}-4 \ln|1+\sqrt{x}|+c$$

$$4. \int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx = 2 \int (1+\sqrt{x})^{\frac{1}{2}} \frac{1}{2\sqrt{x}} dx = \frac{4}{3} (1+\sqrt{x})^{\frac{3}{2}} + c$$

IMPROPER INTEGRALS

Definition (Improper Integrals with a discontinuous integrand):

1. If f is continuous on $[a, b)$ and $|f(x)| \rightarrow \infty$ as $x \rightarrow b^-$ then

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

2. If f is continuous on $(a, b]$ and $|f(x)| \rightarrow \infty$ as $x \rightarrow a^+$ then

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

In either case, if the limit exists (and equals a value L) then the improper integral **converges** (to L). If the limit does not exist then the improper integral **diverges**.

Remark :

If f is continuous on $[a, b]$ except at a point $c \in (a, b)$ and $|f(x)| \rightarrow \infty$ as $x \rightarrow c$

then
$$\int_a^b f(x) dx = \lim_{t \rightarrow c^-} \int_a^t f(x) dx + \lim_{t \rightarrow c^+} \int_t^b f(x) dx$$

If both limits exist (and equals L_1 and L_2 respectively) then the improper integral **converges** (to $L_1 + L_2$). If at least one of the limits does not exist then the improper integral **diverges**.

Definition (Improper Integrals with an infinite limit of integration) :

1. If f is continuous on $[a, \infty)$ then
$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

2. If f is continuous on $(-\infty, a]$ then
$$\int_{-\infty}^a f(x) dx = \lim_{t \rightarrow -\infty} \int_t^a f(x) dx$$

In either case, if the limit exists (and equals a value L) then the improper integral **converges** (to L). If the limit does not exist then the improper integral **diverges**.

Remark :

If f is continuous on $(-\infty, \infty)$ then for any constant a

$$\int_{-\infty}^\infty f(x) dx = \lim_{t \rightarrow -\infty} \int_t^a f(x) dx + \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

If both limits exist (and equals L_1 and L_2 respectively) then the improper integral **converges** (to $L_1 + L_2$). If at least one of the limits does not exist then the improper integral **diverges**.

Examples :

1. $\int_0^{\infty} x e^{-x} dx$

The function $x e^{-x}$ is continuous on $[0, \infty)$

$$\int_0^{\infty} x e^{-x} dx = \lim_{t \rightarrow \infty} \int_0^t x e^{-x} dx$$

Using integration by parts

$$\begin{aligned} u &= x & dv &= e^{-x} dx \\ du &= dx & v &= -e^{-x} \end{aligned}$$

$$\int_0^{\infty} x e^{-x} dx = \lim_{t \rightarrow \infty} \left([-x e^{-x}]_0^t - \int_0^t -e^{-x} dx \right)$$

$$= \lim_{t \rightarrow \infty} \left([-x e^{-x}]_0^t - [e^{-x}]_0^t \right)$$

$$= \lim_{t \rightarrow \infty} \left([(-t e^{-t}) - (0) e^0] - [(e^{-t} - e^0)] \right) = \lim_{t \rightarrow \infty} \left(\frac{-t}{e^t} - e^{-t} + 1 \right)$$

Note that $\lim_{t \rightarrow \infty} \frac{-t}{e^t} = \left(\frac{-\infty}{\infty} \right)$

Apply L'Hôpital's rule

$$\lim_{t \rightarrow \infty} \frac{-t}{e^t} = \lim_{t \rightarrow \infty} \frac{-1}{e^t} = 0$$

Therefore, $\lim_{t \rightarrow \infty} \left(\frac{-t}{e^t} - e^{-t} + 1 \right) = 0 - 0 + 1 = 1$

Hence, $\int_0^{\infty} x e^{-x} dx$ converges to 1 .

2. $\int_1^{\infty} \frac{\ln x}{x} dx$

The function $\frac{\ln x}{x}$ is continuous on $[1, \infty)$

$$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \ln x \frac{1}{x} dx$$

$$= \lim_{t \rightarrow \infty} \left[\frac{(\ln x)^2}{2} \right]_1^t = \lim_{t \rightarrow \infty} \left[\frac{(\ln t)^2}{2} - \frac{\ln(1)}{2} \right] = \lim_{t \rightarrow \infty} \frac{(\ln t)^2}{2} = \infty$$

Therefore, $\int_1^{\infty} \frac{\ln x}{x} dx$ diverges .

3. $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

The function $\frac{1}{1+x^2}$ is continuous on $(-\infty, \infty)$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{1}{1+x^2} dx + \lim_{t \rightarrow \infty} \int_0^t \frac{1}{1+x^2} dx \\
&= \lim_{t \rightarrow -\infty} [\tan^{-1} x]_t^0 + \lim_{t \rightarrow \infty} [\tan^{-1} x]_0^t \\
&= \lim_{t \rightarrow -\infty} [\tan^{-1}(0) - \tan^{-1} t] + \lim_{t \rightarrow \infty} [\tan^{-1} t - \tan^{-1}(0)] \\
&= \tan^{-1}(0) - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - \tan^{-1}(0) = \frac{\pi}{2} + \frac{\pi}{2} = \pi .
\end{aligned}$$

Therefore, $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$ converges to π .

4. $\int_0^1 \frac{x}{(x^2-1)^3} dx$

The function $\frac{x}{(x^2-1)^3}$ is not continuous at $x=1$.

$$\begin{aligned}
\int_0^1 \frac{x}{(x^2-1)^3} dx &= \lim_{t \rightarrow 1^-} \int_0^t \frac{x}{(x^2-1)^3} dx = \lim_{t \rightarrow 1^-} \frac{1}{2} \int_0^t \frac{2x}{(x^2-1)^3} dx \\
&= \lim_{t \rightarrow 1^-} \frac{1}{2} \left[\frac{(x^2-1)^{-2}}{-2} \right]_0^t = \lim_{t \rightarrow 1^-} -\frac{1}{4} \left[\frac{1}{(t^2-1)^2} - \frac{1}{(0-1)^2} \right] \\
&= \lim_{t \rightarrow 1^-} -\frac{1}{4} \left[\frac{1}{(t^2-1)^2} - 1 \right] = -\infty
\end{aligned}$$

Therefore, $\int_0^1 \frac{x}{(x^2-1)^3} dx$ diverges.

5. $\int_1^e \frac{1}{x\sqrt{\ln x}} dx$

The function $\frac{1}{x\sqrt{\ln x}}$ is not continuous at $x=1$

$$\begin{aligned}
\int_1^e \frac{1}{x\sqrt{\ln x}} dx &= \lim_{t \rightarrow 1^+} \int_t^e \frac{1}{x\sqrt{\ln x}} dx = \lim_{t \rightarrow 1^+} \int_t^e (\ln x)^{-\frac{1}{2}} \frac{1}{x} dx \\
&= \lim_{t \rightarrow 1^+} \left[2(\ln x)^{\frac{1}{2}} \right]_t^e = \lim_{t \rightarrow 1^+} 2 \left[\sqrt{\ln(e)} - \sqrt{\ln t} \right] \\
&= \lim_{t \rightarrow 1^+} 2 \left[1 - \sqrt{\ln t} \right] = 2[1-0] = 2
\end{aligned}$$

Therefore, $\int_1^e \frac{1}{x\sqrt{\ln x}} dx$ converges to 2.

6. $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}} dx$

The function $\frac{1}{x\sqrt{x^2-1}}$ is not continuous at $x=1$.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}} dx &= \int_1^2 \frac{1}{x\sqrt{x^2-1}} dx + \int_2^{\infty} \frac{1}{x\sqrt{x^2-1}} dx \\
&= \lim_{t \rightarrow 1^+} \int_t^2 \frac{1}{x\sqrt{x^2-1}} dx + \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x\sqrt{x^2-1}} dx \\
&= \lim_{t \rightarrow 1^+} [\sec^{-1} x]_t^2 + \lim_{t \rightarrow \infty} [\sec^{-1} x]_2^t \\
&= \lim_{t \rightarrow 1^+} [\sec^{-1}(2) - \sec^{-1} t] + \lim_{t \rightarrow \infty} [\sec^{-1} t - \sec^{-1}(2)] \\
&= \sec^{-1}(2) - 0 + \frac{\pi}{2} - \sec^{-1}(2) = \frac{\pi}{2} .
\end{aligned}$$

Therefore, $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}} dx$ converges to $\frac{\pi}{2}$.

Exercises :

Determine whether the following improper integrals converge or diverge

1. $\int_{-\infty}^0 e^x dx$

2. $\int_0^8 \frac{1}{\sqrt[3]{x}} dx$

Hint : $\frac{1}{\sqrt[3]{x}}$ is not continuous at $x = 0$

3. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx$

Hint : $\frac{\cos x}{\sqrt{\sin x}}$ is not continuous at $x = 0$

4. Show that the improper integral $\int_0^1 x \ln x dx$ converges .

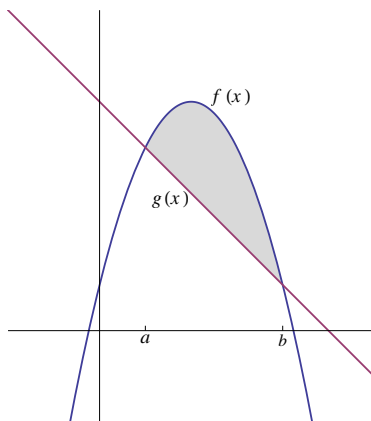
5. $\int_0^1 \frac{1}{\sqrt{2x-x^2}} dx$

Hint : $\frac{1}{\sqrt{2x-x^2}}$ is not continuous at $x = 0$, complete the square

6. $\int_0^{\infty} \frac{1}{x^2} dx$

Hint : $\frac{1}{x^2}$ is not continuous at $x = 0$

AREA BETWEEN CURVES



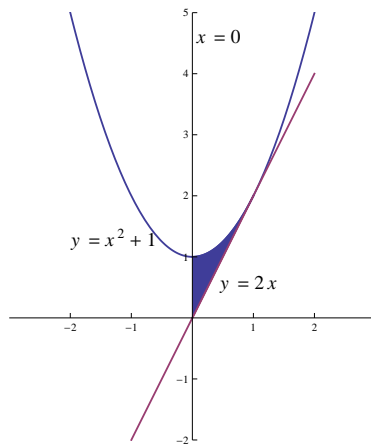
In the above figure the graphs of $f(x)$ and $g(x)$ intersect at the points $x = a$ and $x = b$.

The area bounded by the graphs of the curves of $f(x)$ and $g(x)$ equals

$$\int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b [f(x) - g(x)] \, dx$$

Examples :

1. Find the area bounded by the graphs of the curves of $y = x^2 + 1$, $y = 2x$ and $x = 0$.



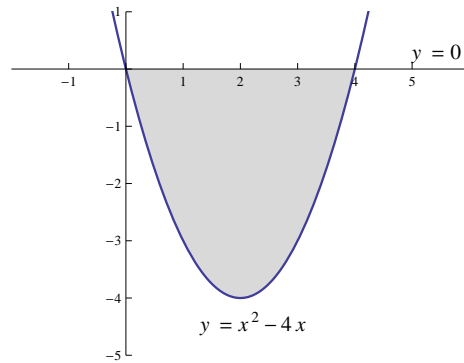
Note that $y = x^2 + 1$ is a parabola opens upward with vertex $(0, 1)$, $y = 2x$ is a straight line passing through the origin and $x = 0$ is the y-axis.

Points of intersection between $y = x^2 + 1$ and $y = 2x$ is :

$$x^2 + 1 = 2x \Rightarrow x^2 - 2x + 1 = 0 \Rightarrow (x - 1)^2 = 0 \Rightarrow x = 1$$

$$\begin{aligned} \text{The desired area} &= \int_0^1 [(x^2 - 1) - 2x] dx = \int_0^1 (x - 1)^2 dx \\ &= \left[\frac{(x - 1)^3}{3} \right]_0^1 = \frac{(1 - 1)^3}{3} - \frac{(0 - 1)^3}{3} = \frac{1}{3}. \end{aligned}$$

2. Find the area bounded by the graphs of the curves of $y = x^2 - 4x$ and $y = 0$



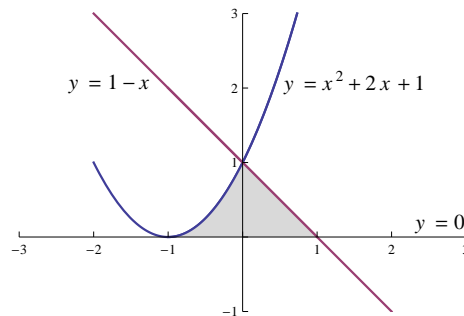
Note that $x^2 - 4x = (x^2 - 4x + 4) - 4 = (x - 2)^2 - 4$ is a parabola opens upward with vertex $(2, -4)$ and $y = 0$ is the x-axis .

Points of intersection between $y = x^2 - 4x$ and $y = 0$

$$x^2 - 4x = 0 \Rightarrow x(x - 4) = 0 \Rightarrow x = 0, \quad x = 4.$$

$$\begin{aligned} \text{The desired area} &= \int_0^4 [0 - (x^2 - 4x)] dx = \int_0^4 (4x - x^2) dx = \left[2x^2 - \frac{x^3}{3} \right]_0^4 \\ &= \left[\left(2(4)^2 - \frac{(4)^3}{3} \right) - 0 \right] = 32 - \frac{64}{3} = \frac{96 - 64}{3} = \frac{32}{3}. \end{aligned}$$

3. Find the area bounded by the graphs of the curves of $y = x^2 + 2x + 1$, $y = 1 - x$ and $y = 0$.



Note that $y = x^2 + 2x + 1 = (x + 1)^2$ is a parabola opens upward with vertex $(-1, 0)$, $y = 1 - x$ is a straight line and $y = 0$ is the x-axis .

Points of intersection between $y = x^2 + 2x + 1$ and $y = 1 - x$

$$x^2 + 2x + 1 = 1 - x \Rightarrow x^2 + 3x = 0 \Rightarrow x(x + 3) = 0 \Rightarrow x = 0, \quad x = -3.$$

Points of intersection between $y = x^2 + 2x + 1$ and $y = 0$ is $x = -1$.

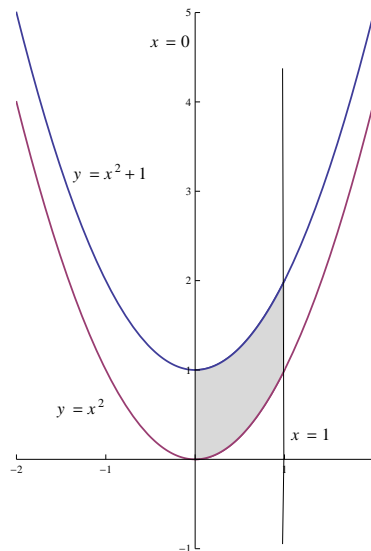
Points of intersection between $y = 1 - x$ and $y = 0$ is $x = 1$.

$$\text{The desired area} = \int_{-1}^0 (x^2 + 2x + 1) dx + \int_0^1 (1 - x) dx$$

$$= \left[\frac{(x+1)^3}{3} \right]_{-1}^0 + \left[x - \frac{x^2}{2} \right]_0^1 = \left[\frac{(0+1)^3}{3} - \frac{(-1+1)^3}{3} \right] + \left[\left(1 - \frac{(1)^2}{2} \right) - 0 \right]$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

4. Find the area bounded by the graphs of the curves of $y = x^2$, $y = x^2 + 1$, $x = 0$ and $x = 1$.

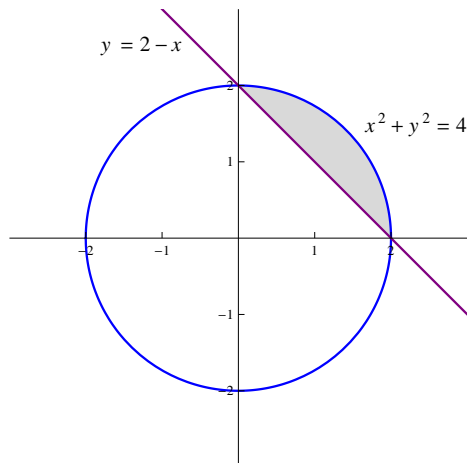


Note that $x^2 + 1$ is a parabola opens upward with vertex $(0, 1)$, $y = x^2$ is another parabola opens upward with vertex $(0, 0)$, $x = 0$ is the y-axis and $x = 1$ is a straight line parallel to the y-axis and passing through the point $(1, 0)$.

Note also that $y = x^2 + 1$ and $y = x^2$ do not intersect.

$$\text{The desired area} = \int_0^1 [(x^2 + 1) - x^2] dx = \int_0^1 dx = [x]_0^1 = 1 - 0 = 1$$

5. Find the area inside the graph of the curve $x^2 + y^2 = 4$ and above $x + y = 2$.



NOTE : The desired area is one fourth of the area of the circle minus the area of the triangle which equals to $\pi - 2$

Note that $x^2 + y^2 = 4$ is a circle with center $= (0, 0)$ and radius $= 2$ and $y = 2 - x$ is a straight line.

Points of intersection between $x^2 + y^2 = 4$ and $y = 2 - x$

$$x^2 + (2 - x)^2 = 4 \Rightarrow x^2 + 4 - 4x + x^2 = 4 \Rightarrow 2x^2 - 4x = 0$$

$$\Rightarrow x^2 - 2x = 0 \Rightarrow x(x - 2) = 0 \Rightarrow x = 0, x = 2$$

Note also that $x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$, where $\sqrt{4 - x^2}$ represents the upper half of the circle and $-\sqrt{4 - x^2}$ represents the lower half of the circle.

$$\text{The desired area} = \int_0^2 \sqrt{4 - x^2} dx - \int_0^2 (2 - x) dx = I_1 - I_2$$

$$I_1 = \int_0^2 \sqrt{4 - x^2} dx$$

$$\text{Put } x = 2 \sin \theta \Rightarrow dx = 2 \cos \theta d\theta$$

$$\text{If } x = 0 \Rightarrow 2 \sin \theta = 0 \Rightarrow \sin \theta = 0 \Rightarrow \theta = 0$$

$$\text{If } x = 2 \Rightarrow 2 \sin \theta = 2 \Rightarrow \sin \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sqrt{4 - 4 \sin^2} 2 \cos \theta d\theta = \int_0^{\frac{\pi}{2}} 4 \cos^2 \theta d\theta$$

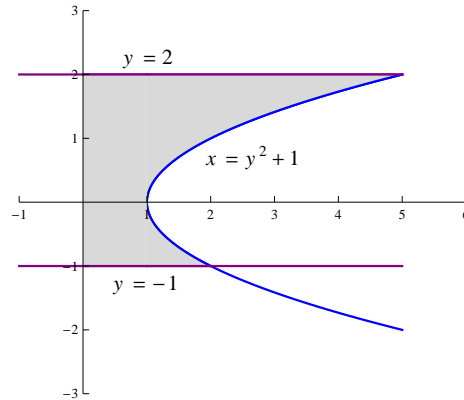
$$= 4 \int_0^{\frac{\pi}{2}} \frac{1}{2} [1 + \cos 2\theta] d\theta = 2 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}}$$

$$= 2 \left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left(0 + \frac{\sin(0)}{2} \right) \right] = 2 \left[\left(\frac{\pi}{2} + 0 \right) - (0 + 0) \right] = 2 \frac{\pi}{2} = \pi$$

$$I_2 = \int_0^2 (2-x) dx = \left[2x - \frac{x^2}{2} \right]_0^2 = \left[\left(2(2) - \frac{2^2}{2} \right) - (0-0) \right] = 4-2 = 2$$

Hence , The desired area = $I_1 - I_2 = \pi - 2$.

6. Find the area bounded by the graphs of the curves of $x = y^2 + 1$, $x = 0$, $y = -1$ and $y = 2$.

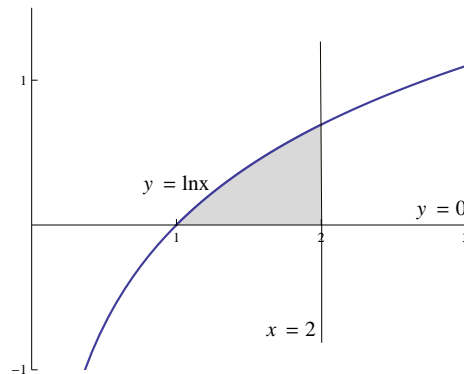


Note that $x = y^2 + 1$ is a parabola opens to the right with vertex $(1, 0)$, $x = 0$ is the y -axis , $y = 2$ is a straight line parallel to the x -axis and passing through the point $(0, 2)$ also $y = -1$ is another straight line parallel to the x -axis and passing through the point $(0, -1)$.

$$\begin{aligned} \text{The desired area} &= \int_{-1}^2 (y^2 + 1) dy = \left[\frac{y^3}{3} + y \right]_{-1}^2 \\ &= \left[\left(\frac{(2)^3}{3} + 2 \right) - \left(\frac{(-1)^3}{3} + (-1) \right) \right] = \frac{8}{3} + 2 + \frac{1}{3} + 1 = \frac{18}{3} = 6 \end{aligned}$$

Examples : Set up integrals to evaluate the areas bounded by the graphs of the curves of :

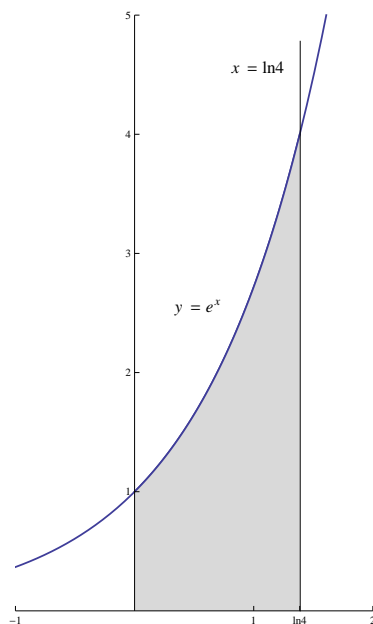
1. $y = \ln x$, $y = 0$ and $x = 2$.



Note that $y = \ln x$ intersects the x-axis at $x = 1$

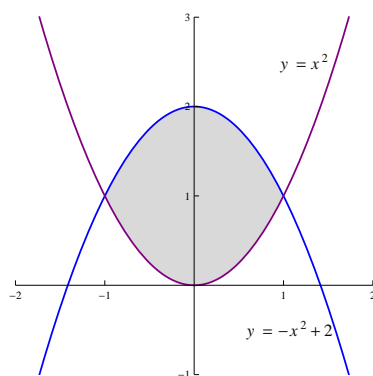
The desired area = $\int_1^2 \ln x \, dx$

2. $y = e^x$, $x = \ln 4$, $x = 0$ and $y = 0$.



The desired area = $\int_0^{\ln 4} e^x \, dx$

3. $y = x^2$ and $y = -x^2 + 2$



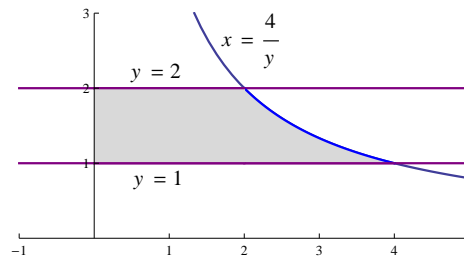
Note that $y = x^2$ is a parabola opens upward with vertex $(0,0)$ and $y = -x^2 + 2$ is another parabola opens downward with vertex $(0,2)$

Points of intersection between $y = x^2$ and $y = -x^2 + 2$

$$x^2 = -x^2 + 2 \Rightarrow 2x^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

The desired area = $\int_{-1}^1 [(-x^2 + 2) - x^2] dx$

4. $y = \frac{4}{x}$, $x = 0$, $y = 1$ and $y = 2$.

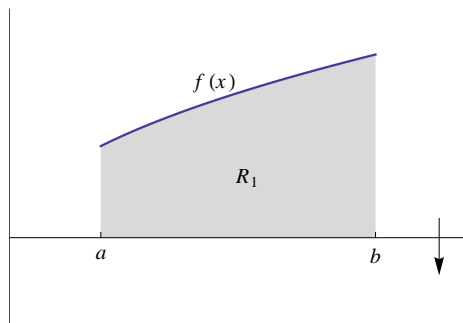


The desired area = $\int_1^2 \frac{4}{y} dy$

VOLUME OF A SOLID OF REVOLUTION Disk or Washer method

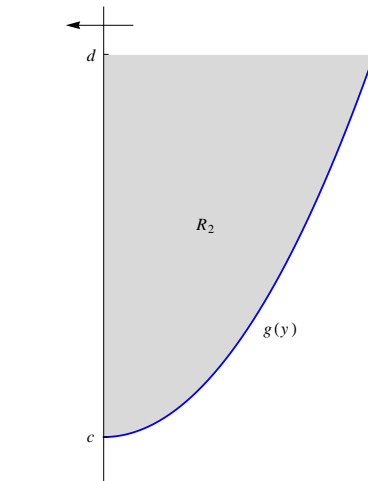
1. Disk Method

Recall that the volume of a right circular cylinder equals $\pi r^2 h$ where r is the radius of the base (which is a circle) and h is the height of the cylinder .



In the above figure R_1 is the region bounded by the graphs of the curves of $f(x)$, $x = a$, $x = b$ and the x-axis.

Using disk method , the volume of the solid of revolution generated by revolving the region R_1 around the x-axis is $V = \pi \int_a^b [f(x)]^2 dx$

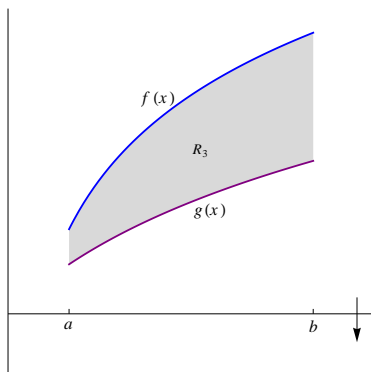


In the above figure R_2 is the region bounded by the graphs of the curves of $g(y)$, $y = d$ and the y-axis.

Using disk method , the volume of the solid of revolution generated by revolving the region R_2 around the y-axis is $V = \pi \int_c^d [g(y)]^2 dy$

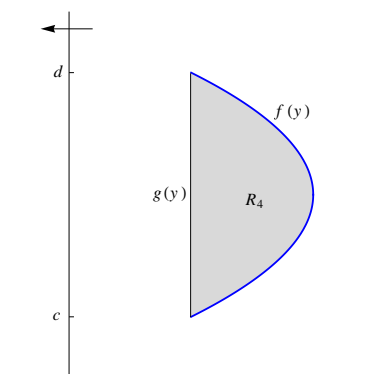
2. Washer Method

Volume of a washer = $\pi [(outer\ radius)^2 - (inner\ radius)^2]$ (thickness)



In the above figure R_3 is the region bounded by the graphs of the curves of $f(x)$, $g(x)$, $x = a$ and $x = b$.

Using washer method, the volume of the solid of revolution generated by revolving the region R_3 around the x-axis is $V = \pi \int_a^b [(f(x))^2 - (g(x))^2] dx$

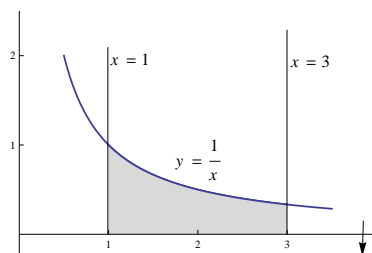


In the above figure R_4 is the region bounded by the graphs of the curves of $f(y)$ and $g(y)$, where $f(y)$ and $g(y)$ intersect at the points $y = c$ and $y = d$.

Using washer method, the volume of the solid of revolution generated by revolving the region R_4 around the y-axis is $V = \pi \int_c^d [(f(y))^2 - (g(y))^2] dy$

Examples : Use disk or washer method to find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the curves of :

1. $y = \frac{1}{x}$, $x = 1$, $x = 3$ and $y = 0$, around the x-axis.

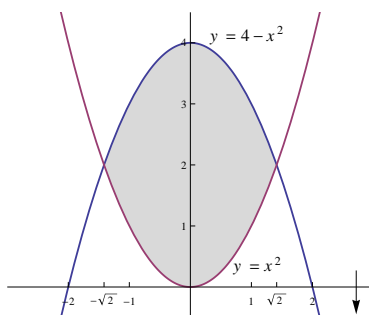


Using Disk Method

$$V = \pi \int_1^3 \left(\frac{1}{x}\right)^2 dx = \pi \int_1^3 x^{-2} dx.$$

$$V = \pi \left[-\frac{1}{x}\right]_1^3 = \pi \left[-\frac{1}{3} + 1\right] = \frac{2}{3}\pi$$

2. $y = x^2$ and $y = 4 - x^2$, around the x-axis .



Note that $y = x^2$ is a parabola opens upward with vertex $(0,0)$ and $y = 4 - x^2$ is a parabola opens downward with vertex $(0,4)$.

Points of intersection between $y = x^2$ and $y = 4 - x^2$:

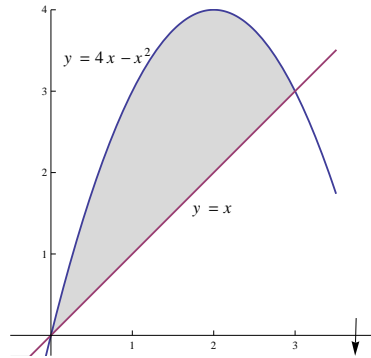
$$x^2 = 4 - x^2 \Rightarrow 2x^2 = 4 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

Using Washer Method

$$V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} [(4 - x^2)^2 - (x^2)^2] dx = 2\pi \int_0^{\sqrt{2}} [16 - 8x^2 + x^4 - x^4] dx$$

$$= 2\pi \int_0^{\sqrt{2}} (16 - 8x^2) dx = 2\pi \left[16x - \frac{8}{3}x^3\right]_0^{\sqrt{2}} = \frac{64\sqrt{2}}{3}\pi$$

3. $y = 4x - x^2$ and $y = x$, around the x-axis .



$4x - x^2 = -(x^2 - 4x + 4) + 4 = 4 - (x - 2)^2$ is a parabola opens downward with vertex $(2, 4)$ and $y = x$ is a straight line passing through the origin.

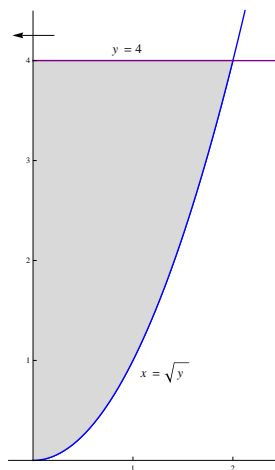
Points of intersection between $y = 4x - x^2$ and $y = x$

$$x = 4x - x^2 \Rightarrow x^2 - 3x = 0 \Rightarrow x(x - 3) = 0 \Rightarrow x = 0, x = 3$$

Using Washer Method

$$\begin{aligned} V &= \pi \int_0^3 [(4x - x^2)^2 - (x)^2] dx = \pi \int_0^3 [16x^2 - 8x^3 + x^4 - x^2] dx \\ &= \pi \int_0^3 [x^4 - 8x^3 + 15x^2] dx = \pi \left[\frac{x^5}{5} - 2x^4 + 5x^3 \right]_0^3 = \frac{108}{5} \pi \end{aligned}$$

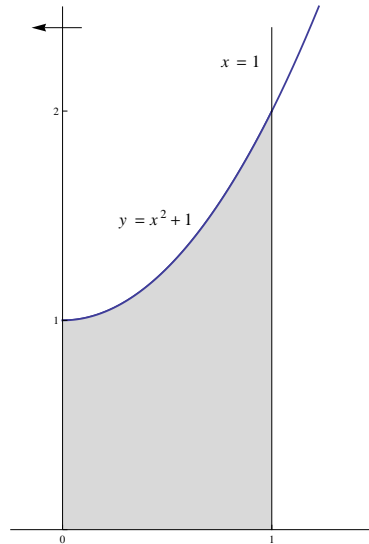
4. $x = \sqrt{y}$, $x = 0$ and $y = 4$, around the y-axis



Using Disk Method

$$V = \pi \int_0^4 (\sqrt{y})^2 dy = \pi \int_0^4 y dy = \pi \left[\frac{y^2}{2} \right]_0^4 = 8\pi$$

5. $y = x^2 + 1$, $y = 0$, $x = 0$ and $x = 1$, around the y-axis .



Note that $y = x^2 + 1$ is a parabola opens upward with vertex $(0, 1)$, $x = 1$ is a straight line parallel to the y-axis and passing through the point $(1, 0)$

Point of intersection between $y = x^2 + 1$ and $x = 1$ is $(1, 2)$.

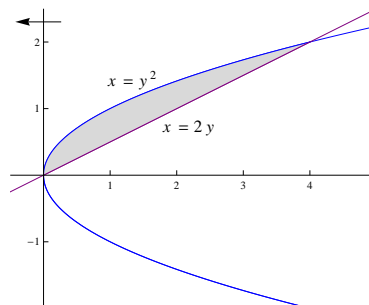
$y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \pm\sqrt{y-1}$, where $x = \sqrt{y-1}$ is the right half of the parabola and $y = -\sqrt{y-1}$ is the left half of the parabola .

Using Washer Method

$$V = \pi \int_0^2 (1)^2 dy - \pi \int_1^2 (\sqrt{y-1})^2 dy$$

$$V = \pi [y]_0^2 - \pi \left[\frac{y^2}{2} - y \right]_1^2 = \frac{3}{2}\pi$$

6. $x = y^2$ and $x = 2y$, around the y-axis .



Note that $x = y^2$ is a parabola opens to the right with vertex $(0, 0)$ and $x = 2y$ is a straight line passing through the origin.

Points of intersection between $x = y^2$ and $x = 2y$

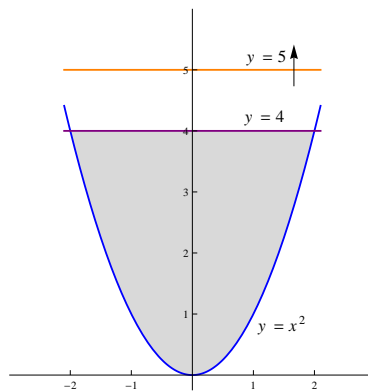
$$y^2 = 2y \Rightarrow y^2 - 2y = 0 \Rightarrow y(y - 2) = 0 \Rightarrow y = 0, y = 2$$

Using Washer Method

$$V = \pi \int_0^2 [(2y)^2 - (y^2)^2] dy = \pi \int_0^2 (4y^2 - y^4) dy$$

$$V = \pi \left[\frac{4y^3}{3} - \frac{y^5}{5} \right]_0^2 = \frac{64}{15}\pi$$

7. $y = x^2$ and $y = 4$, around the line $y = 5$.



Note that $y = x^2$ is a parabola opens upward with vertex $(0, 0)$ and $y = 4$ is a straight line parallel to the x -axis and passing through $(0, 4)$.

Points of intersection between $y = x^2$ and $y = 4$

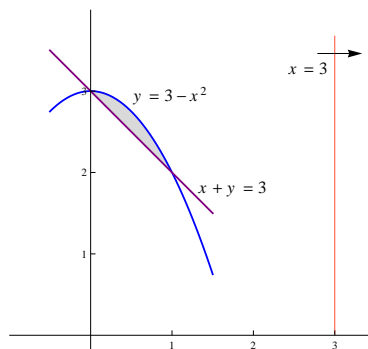
$$x^2 = 4 \Rightarrow x = \pm 2$$

Using Washer Method

$$V = \pi \int_{-2}^2 [(5 - x^2)^2 - (5 - 4)^2] dx = \pi \int_{-2}^2 (24 - 10x^2 + x^4) dx$$

$$V = \pi \left[24x - \frac{10x^3}{3} + \frac{x^5}{5} \right]_{-2}^2 = \frac{832}{15}\pi$$

8. $y + x^2 = 3$ and $y + x = 3$, around the line $x = 3$



Note that $y = 3 - x^2$ is a parabola opens downward with vertex $(0, 3)$ and $x + y = 3$ is a straight line.

Points of intersection between $y + x^2 = 3$ and $x + y = 3$

$$y + x^2 = x + y \Rightarrow x^2 - x = 0 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0, x = 1$$

$$\Rightarrow y = 2, y = 3$$

$y + x^2 = 3 \Rightarrow x^2 = 3 - y \Rightarrow x = \pm\sqrt{3 - y}$, where $x = \sqrt{3 - y}$ is the right half of the parabola and $x = -\sqrt{3 - y}$ is the left half of the parabola.

Using Washer Method

$$V = \pi \int_2^3 \left[(3 - (3 - y))^2 - (3 - \sqrt{3 - y})^2 \right] dy$$

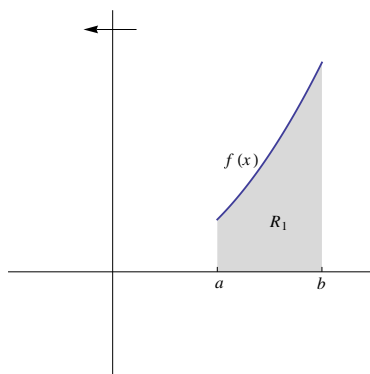
$$= \pi \int_2^3 \left[y^2 - (9 - 6\sqrt{3 - y} + 3 - y) \right] dy$$

$$= \pi \int_2^3 (y^2 + y + 6\sqrt{3 - y} - 12) dy$$

$$V = \pi \left[\frac{y^3}{3} + \frac{y^2}{2} - 4(3 - y)^{\frac{3}{2}} - 12y \right]_2^3 = \frac{5}{6}\pi$$

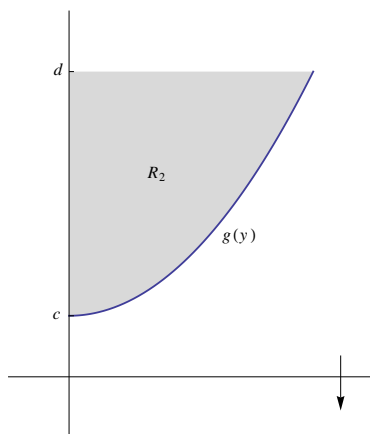
VOLUME OF A SOLID OF REVOLUTION
Cylindrical shells method

Volume of a shell = 2π (average radius) (altitude) (thickness)



In the above figure R_1 is the region bounded by the graphs of the curves of $f(x)$, $x = a$, $x = b$ and the x-axis.

Using cylindrical shells method, the volume of the solid of revolution generated by revolving the region R_1 around the y-axis is $V = 2\pi \int_a^b x f(x) dx$

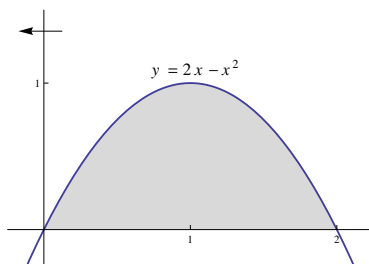


In the above figure R_2 is the region bounded by the graphs of the curves of $g(y)$, $y = d$ and the y-axis.

Using cylindrical shells method, the volume of the solid of revolution generated by revolving the region R_2 around the x-axis is $V = 2\pi \int_c^d y g(y) dy$

Examples : Use cylindrical shells method to find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the curves of :

1. $y = 2x - x^2$ and $y = 0$, around the y-axis .



$y = 2x - x^2 = -(x^2 - 2x + 1) + 1 = 1 - (x - 1)^2$ is a parabola opens downward with vertex $(1, 1)$

Points of intersection between $y = 2x - x^2$ and $y = 0$

$$2x - x^2 = 0 \Rightarrow x(2 - x) = 0 \Rightarrow x = 0 , x = 2$$

Using Cylindrical shells method

$$V = 2\pi \int_0^2 x(2x - x^2) dx = 2\pi \int_0^2 (2x^2 - x^3) dx$$

$$V = 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2 = \frac{8}{3}\pi$$

2. $y = \cos x$, $y = 2x + 1$ and $x = \frac{\pi}{2}$, around the y-axis .

Recall that $\cos(0) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$.

The line $y = 2x + 1$ passes through the point $(0, 1)$.

The desired region is under the line $y = 2x + 1$ and above the curve of $y = \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$

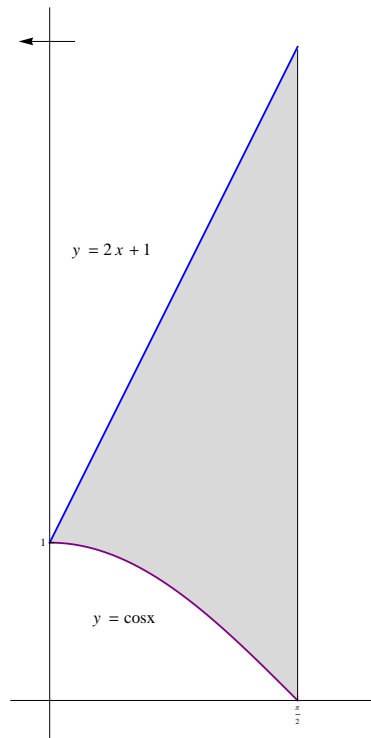
Using Cylindrical shells method

$$V = 2\pi \int_0^{\frac{\pi}{2}} x[(2x + 1) - \cos x] dx$$

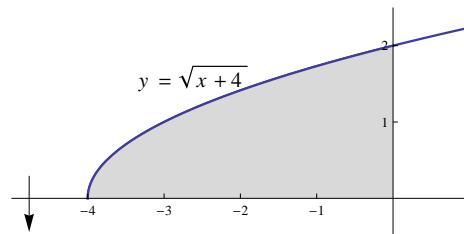
$$V = 2\pi \int_0^{\frac{\pi}{2}} (2x^2 + x) dx - 2\pi \int_0^{\frac{\pi}{2}} (x \cos x) dx$$

$$V = 2\pi \left[\frac{2x^3}{3} + \frac{x^2}{2} \right]_0^{\frac{\pi}{2}} - 2\pi [x \sin x + \cos x]_0^{\frac{\pi}{2}}$$

$$V = 2\pi \left(\frac{\pi^3}{12} + \frac{\pi^2}{8} \right) - 2\pi \left(\frac{\pi}{2} - 1 \right)$$



3. $y = \sqrt{x+4}$, $y = 0$ and $x = 0$, around the x-axis .



$y = \sqrt{x+4}$ is the upper half of the parabola $x = y^2 - 4$ which opens to the right with vertex $(-4, 0)$.

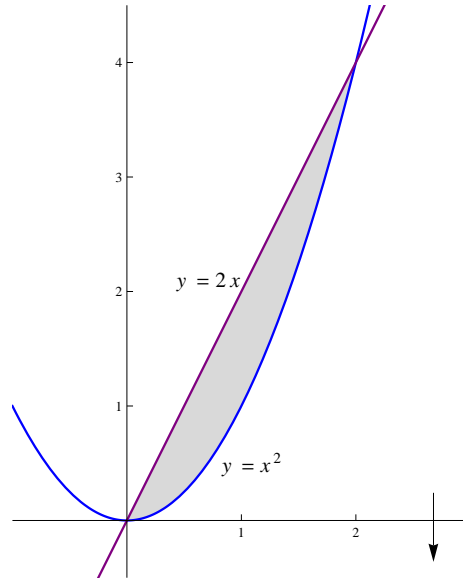
$y = \sqrt{x+4}$ intersects the x-axis at the point $(-4, 0)$ and intersects the y-axis at $(0, 2)$

Using Cylindrical shells method

$$V = 2\pi \int_0^2 y[-(y^2 - 4)] dy = 2 \int_0^2 (4y - y^3) dy$$

$$V = 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2 = 8\pi$$

4. $y = x^2$ and $y = 2x$, around the x-axis .



$y = x^2$ is a parabola open upward with vertex $(0,0)$ and $y = 2x$ is a straight line passing through the origin.

Points of intersection between $y = x^2$ and $y = 2x$

$$x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x - 2) = 0 \Rightarrow x = 0, x = 2$$

$$\Rightarrow y = 0, y = 4$$

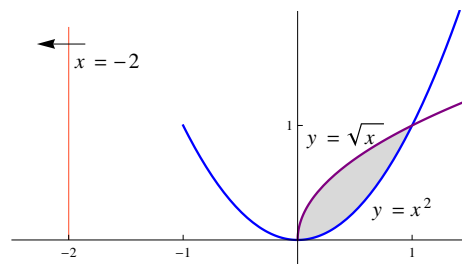
$y = x^2 \Rightarrow x = \pm\sqrt{y}$, where $x = \sqrt{y}$ is the right half of the parabola $y = x^2$ and $x = -\sqrt{y}$ is the left half of the parabola.

Using Cylindrical shells method

$$V = 2\pi \int_0^4 y \left(\sqrt{y} - \frac{y}{2} \right) dy = 2\pi \int_0^4 \left(y^{\frac{3}{2}} - \frac{y^2}{2} \right) dy$$

$$V = 2\pi \left[\frac{2y^{\frac{5}{2}}}{5} - \frac{y^3}{6} \right]_0^4 = \frac{64}{15}\pi$$

5. $y = \sqrt{x}$ and $y = x^2$, around the line $x = -2$.



$y = x^2$ is a parabola opens upward with vertex $(0, 0)$, and $y = \sqrt{x}$ is the upper half of the parabola $x = y^2$.

Points of intersection between $y = x^2$ and $y = \sqrt{x}$

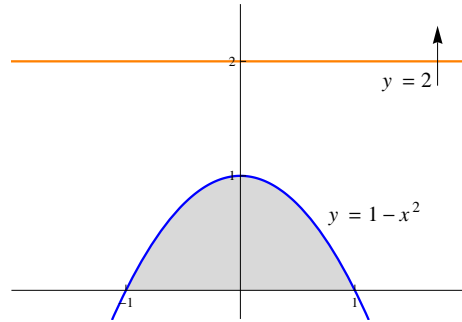
$$x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0, x = 1$$

Using Cylindrical shells method

$$V = 2\pi \int_0^1 (x+2)(\sqrt{x} - x^2) dx = 2\pi \int_0^1 (-x^3 - 2x^2 + x^{\frac{3}{2}} + 2x^{\frac{1}{2}}) dx$$

$$V = 2\pi \left[-\frac{x^4}{4} - \frac{2x^3}{3} + \frac{2x^{\frac{5}{2}}}{5} + \frac{x^{\frac{3}{2}}}{3} \right]_0^1 = \frac{49}{30}\pi$$

6. $y = 1 - x^2$ and $y = 0$, around the line $y = 2$.



$y = 1 - x^2$ is a parabola opens downward with vertex $(0, 1)$ and $y = 0$ is the x-axis.

$y = 1 - x^2$ intersects $y = 0$ at $x = \pm 1$.

$y = 1 - x^2 \Rightarrow x^2 = 1 - y \Rightarrow x = \pm\sqrt{1 - y}$, where $y = \sqrt{1 - y}$ represents the right half of the parabola and $y = -\sqrt{1 - y}$ represents the left half.

Note that the region is symmetric with respect to the y-axis.

Using Cylindrical shells method

$$V = 2 \left(2\pi \int_0^1 (2 - y)\sqrt{1 - y} dy \right)$$

Put $u^2 = 1 - y$ then $2u du = -dy$

If $y = 0$ then $u = 1$, and if $y = 1$ then $u = 0$

$$V = 4\pi \int_1^0 (2 + u^2 - 1) u (-2u) du = 4\pi \int_0^1 (u^2 + 1)2u^2 du$$

$$V = 4\pi \int_0^1 (2u^4 + 2u^2) du = 4\pi \left[\frac{2u^5}{5} + \frac{2u^3}{3} \right]_0^1 = \frac{64}{15}\pi$$

ARC LENGTH

If $f(x)$ is continuous function on the interval $[a, b]$, then the arc length of $f(x)$ from $x = a$ to $x = b$ is $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$

If $g(y)$ is continuous function on the interval $[c, d]$, then the arc length of $g(y)$ from $y = c$ to $y = d$ is $L = \int_c^d \sqrt{1 + [g'(y)]^2} dy$

Examples : Find the arc length of the following :

1. $y = \frac{x^3}{12} + \frac{1}{x}$ from $A = (1, \frac{13}{12})$ to $B = (2, \frac{7}{6})$.

$$f(x) = \frac{x^3}{12} + \frac{1}{x} \Rightarrow f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2} dx = \int_1^2 \sqrt{1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}} dx \\ &= \int_1^2 \sqrt{\frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}} dx = \int_1^2 \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2} dx = \int_1^2 \left|\frac{x^2}{4} + \frac{1}{x^2}\right| dx \end{aligned}$$

$$L = \int_1^2 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx = \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^2 = \frac{13}{12}$$

2. $y = \frac{1}{2}(e^x + e^{-x})$, $x \in [0, 2]$

$$f(x) = \frac{e^x + e^{-x}}{2} = \cosh x \Rightarrow f'(x) = \sinh x$$

$$L = \int_0^2 \sqrt{1 + \sinh^2 x} dx = \int_0^2 \sqrt{\cosh^2 x} dx$$

$$= \int_0^2 |\cosh x| dx = \int_0^2 \cosh x dx$$

$$L = [\sinh x]_0^2 = \sinh(2) - \sinh(0) = \frac{e^2 - e^{-2}}{2} - 0 = \frac{e^2 - e^{-2}}{2}$$

3. $x^2 + y^2 = 25$, $-5 \leq y \leq 5$

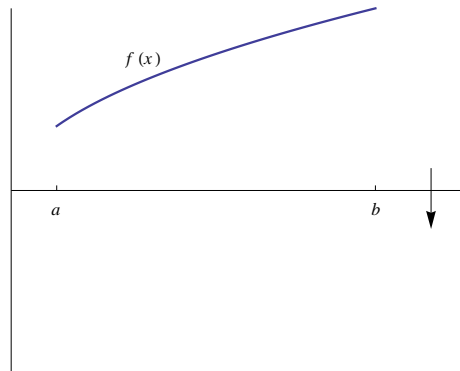
Note : In this problem the arc length is equal to half of the perimeter of the circle $x^2 + y^2 = 25$, the arc length is equal to 5π .

$$x^2 + y^2 = 25 \Rightarrow x^2 = 25 - y^2 \Rightarrow x = \pm\sqrt{25 - y^2}, \text{ in this problem } x = \sqrt{25 - y^2}$$

$$g(y) = \sqrt{25 - y^2} \Rightarrow g'(y) = \frac{-y}{\sqrt{25 - y^2}}$$

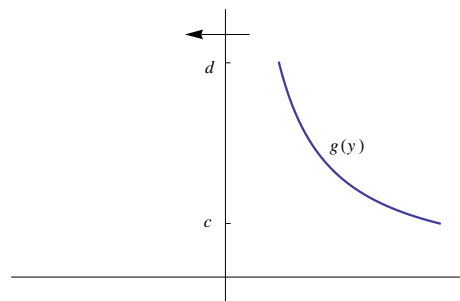
$$\begin{aligned} L &= \int_{-5}^5 \sqrt{1 + \left(\frac{-y}{\sqrt{25-y^2}}\right)^2} dy = \int_{-5}^5 \sqrt{1 + \frac{y^2}{25-y^2}} dy \\ &= \int_{-5}^5 \sqrt{\frac{25-y^2+y^2}{25-y^2}} dy = 5 \int_{-5}^5 \frac{1}{\sqrt{25-y^2}} dy \\ L &= 5 \left[\sin^{-1} \left(\frac{y}{5} \right) \right]_{-5}^5 = 5 [\sin^{-1}(1) - \sin^{-1}(-1)] \\ &= 5 \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] = 5\pi . \end{aligned}$$

SURFACE AREA
(SURFACE OF REVOLUTION)



If $f(x)$ is a continuous function on the interval $[a, b]$, then the surface area generated by revolving the graph of the function $f(x)$ around the x-axis is

$$SA = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx$$

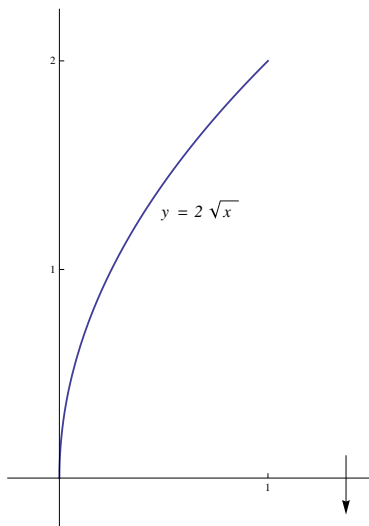


If $g(y)$ is a continuous function on the interval $[c, d]$, then the surface area generated by revolving the graph of the function $g(y)$ around the y-axis is

$$SA = 2\pi \int_c^d g(y) \sqrt{1 + [g'(y)]^2} dy$$

Examples : Find the surface area generated by revolving the following functions around the given axis :

1. $4x = y^2$, from $A = (0, 0)$ to $B = (1, 2)$, around the x-axis .



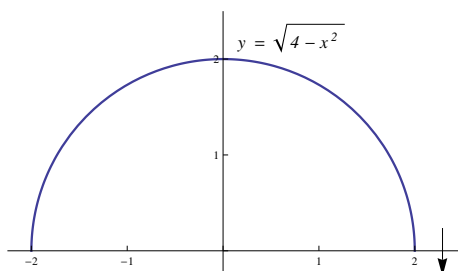
$$4x = y^2 \Rightarrow y = \pm 2\sqrt{x}$$

$$f(x) = 2\sqrt{x} \Rightarrow f'(x) = \frac{1}{\sqrt{x}}$$

$$SA = 2\pi \int_0^1 2\sqrt{x} \sqrt{1 + \left[\frac{1}{\sqrt{x}}\right]^2} dx = 4\pi \int_0^1 \sqrt{x} \sqrt{1 + \frac{1}{x}} dx$$

$$SA = 4\pi \int_0^1 \sqrt{x+1} dx = 4\pi \left[2\frac{(x+1)^{\frac{3}{2}}}{3} \right]_0^1 = \frac{8\pi}{3} (2\sqrt{2} - 1)$$

2. $y = \sqrt{4-x^2}$, $x \in [-2, 2]$, around the x-axis .



Note : It is the surface area of the sphere with radius 2 , and it is equal to $4\pi(2)^2 = 16\pi$

$$f(x) = \sqrt{4-x^2} \Rightarrow f'(x) = \frac{-x}{\sqrt{4-x^2}}$$

$$\begin{aligned}
SA &= 2\pi \int_{-2}^2 \sqrt{4-x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2} dx \\
&= 2\pi \int_{-2}^2 \sqrt{4-x^2} \sqrt{\frac{(4-x^2)+x^2}{4-x^2}} dx = 2\pi \int_{-2}^2 \sqrt{4-x^2} \frac{2}{\sqrt{4-x^2}} dx \\
SA &= 4\pi \int_{-2}^2 dx = 4\pi [x]_{-2}^2 = 16\pi
\end{aligned}$$

3. $y = 2\sqrt[3]{x}$, from $A = (1, 2)$ to $B = (8, 4)$, around the y-axis.

$$y = 2\sqrt[3]{x} \Rightarrow \sqrt[3]{x} = \frac{y}{2} \Rightarrow x = \frac{y^3}{8}$$

$$g(y) = \frac{y^3}{8} \Rightarrow g'(y) = \frac{3}{8}y^2$$

$$SA = 2\pi \int_2^4 \frac{y^3}{8} \sqrt{1 + \left(\frac{3}{8}y^2\right)^2} dy = 2\pi \int_2^4 \frac{y^3}{8} \sqrt{1 + \frac{9}{64}y^4} dy$$

$$= 2\pi \frac{1}{8} \frac{16}{9} \int_2^4 \left(1 + \frac{9}{64}y^4\right)^{\frac{1}{2}} \left(\frac{9}{16}y^3\right) dy$$

$$SA = \frac{4\pi}{9} \left[\frac{2\left(1 + \frac{9}{64}y^4\right)^{\frac{3}{2}}}{3} \right]_2^4$$

4. $y = x^2$, $0 \leq x \leq 2$, around the y-axis.

$$y = x^2 \Rightarrow x = \pm\sqrt{y} \Rightarrow x = \sqrt{y}, \text{ since } 0 \leq x \leq 2$$

$$0 \leq x \leq 2 \Rightarrow 0 \leq y \leq 4$$

$$g(y) = \sqrt{y} \Rightarrow g'(y) = \frac{1}{2\sqrt{y}}$$

$$SA = 2\pi \int_0^4 \sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2} dy = 2\pi \int_0^4 \sqrt{y} \sqrt{1 + \frac{1}{4y}} dy$$

$$SA = 2\pi \int_0^4 \sqrt{y + \frac{1}{4}} dy = 2\pi \left[\frac{2\left(y + \frac{1}{4}\right)^{\frac{3}{2}}}{3} \right]_0^4$$

PARAMETRIC EQUATIONS

Parametric equations are used to describe and represent plane curves.

The parameter "t" is used to write x and y as functions of t .

$C : x = x(t), y = y(t); a \leq t \leq b$ is the general form of a parametric curve, where $a, b \in \mathbb{R}$.

Any point on the parametric curve is represented by $P(t) = (x(t), y(t))$.

Notes :

1. If the parametric curve does not intersect itself then it is called a simple curve.
2. If $P(a) = P(b)$ then the parametric curve is called a closed curve.
3. Parametric equation of a curve indicates its orientation (direction of the path).

Examples : Sketch the graph of the following parametric curves :

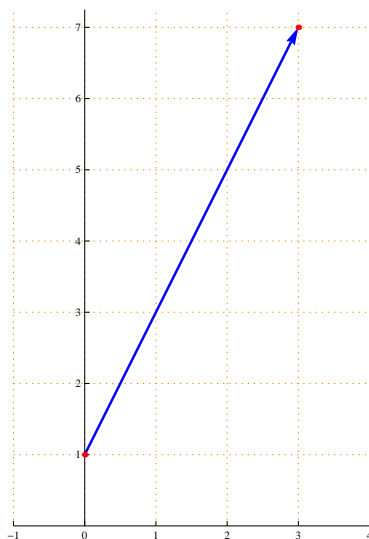
1. $C : x = t + 1, y = 2t + 3; -1 \leq t \leq 2$.

$$x = t + 1 \Rightarrow t = x - 1$$

$$y = 2t + 3 \Rightarrow y = 2(x - 1) + 3 = 2x + 1$$

t	-1	2
x	0	3
y	1	7

The parametric equation represents a line segment from $(0, 1)$ to $(3, 7)$



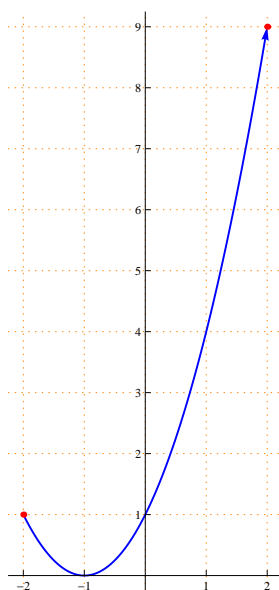
2. $C : x = t - 1, y = t^2 ; -1 \leq t \leq 3$

$$x = t - 1 \Rightarrow t = x + 1$$

$$y = t^2 \Rightarrow y = (x + 1)^2$$

t	-1	3
x	-2	2
y	1	9

The parametric equation represents a part of a parabola from $(-2, 1)$ to $(2, 9)$



3. $C : x = 1 + 3 \cos t, y = -1 + 3 \sin t ; 0 \leq t \leq 2\pi$

$$x = 1 + 3 \cos t \Rightarrow \cos t = \frac{x - 1}{3}$$

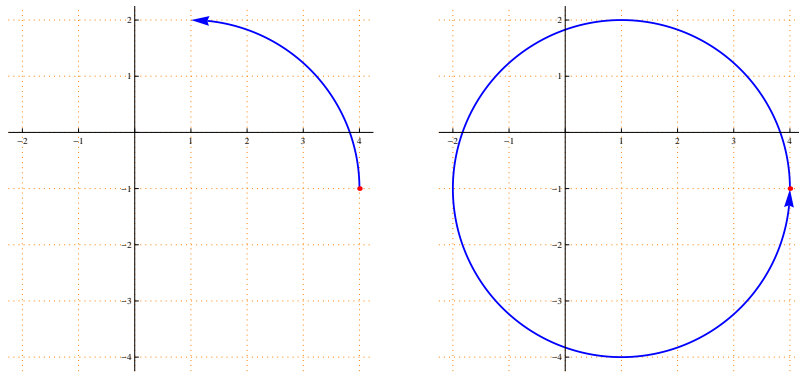
$$y = -1 + 3 \sin t \Rightarrow \sin t = \frac{y + 1}{3}$$

$$\cos^2 t + \sin^2 t = 1 \Rightarrow \frac{(x - 1)^2}{9} + \frac{(y + 1)^2}{9} = 1 \Rightarrow (x - 1)^2 + (y + 1)^2 = 9$$

t	0	$\frac{\pi}{2}$	2π
x	4	1	4
y	-1	2	-1

The parametric equation represents a circle with center $(1, -1)$ and radius $= 3$.

It is a closed curve and its direction is counter-clockwise.



4. $C : x = 3 + 3 \cos t , y = 2 + 2 \sin t ; 0 \leq t \leq 2\pi$

$$x = 3 + 3 \cos t \Rightarrow \cos t = \frac{x - 3}{3}$$

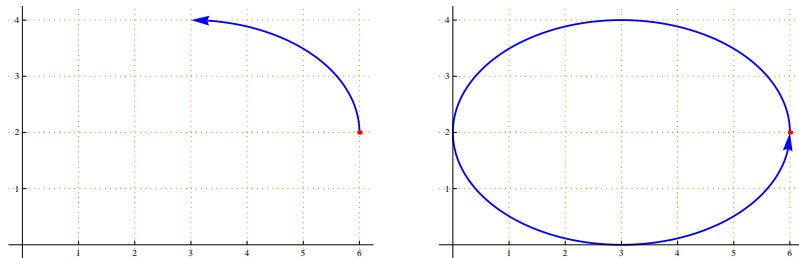
$$y = 2 + 2 \sin t \Rightarrow \sin t = \frac{y - 2}{2}$$

$$\cos^2 t + \sin^2 t = 1 \Rightarrow \frac{(x - 3)^2}{9} + \frac{(y - 2)^2}{4} = 1$$

t	0	$\frac{\pi}{2}$	2π
x	6	3	6
y	2	4	2

The parametric equation represents an ellipse with center $(3, 2)$, the endpoints of the major axis are $(0, 2)$, $(6, 2)$ (its length is 6) and the endpoints of the minor axis are $(3, 0)$, $(3, 4)$ (its length is 4).

it is a closed curve and its direction is counter-clockwise.



The slope of the tangent line to a parametric curve

If $C : x = x(t) , y = y(t) ; a \leq t \leq b$ is a differentiable parametric curve then the slope of the tangent line to C at $t_0 \in [a, b]$ is

$$m = \frac{dy}{dx} \Big|_{t=t_0} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \Big|_{t=t_0}$$

Notes :

1. The tangent line to the parametric curve is horizontal if the slope equals zero , which means that $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.
2. The tangent line to the parametric curve is vertical if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

The second derivative is $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)}$, where $y' = \frac{dy}{dx}$

Examples :

1. The slope of the tangent line to $C : x = t^3 + 1 , y = t^4 - 1$ at $t = 1$ is

(a) $\frac{3}{4}$ (b) 0 (c) $\frac{4}{3}$ (d) None of these

$$\text{Answer : } m = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4t^3}{3t^2}$$

$$\text{The slope at } t = 1 \text{ is } m|_{t=1} = \frac{4}{3}$$

The right answer is (c) .

2. If $C : x = \sqrt{t} , y = \frac{1}{4}(t^2 - 1)$, find the first and second derivatives at $t = 4$.

$$\text{First derivative : } \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{1}{2}t\right)}{\left(\frac{1}{2\sqrt{t}}\right)} = t^{\frac{3}{2}}$$

$$\frac{dy}{dx} \Big|_{t=4} = (4)^{\frac{3}{2}} = 8 .$$

$$\text{Second derivative : } \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{3}{2}t^{\frac{1}{2}}\right)}{\left(\frac{1}{2\sqrt{t}}\right)} = 3t$$

$$\frac{d^2y}{dx^2}\bigg|_{t=4} = 3(4) = 12 .$$

3. If $C : x = 2 \cos t$, $y = 2 \sin t$, find the first and the second derivatives at $t = \frac{\pi}{4}$.

$$\text{First derivative : } \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2 \cos t}{-2 \sin t} = -\cot t$$

$$\frac{dy}{dx}\bigg|_{t=\frac{\pi}{4}} = -\cot\left(\frac{\pi}{4}\right) = -1 .$$

$$\text{Second derivative : } \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\csc^2 t}{-2 \sin t} = \frac{-1}{2 \sin^3 t}$$

$$\frac{d^2y}{dx^2}\bigg|_{t=\frac{\pi}{4}} = \frac{-1}{2\left(\frac{1}{\sqrt{2}}\right)^3} = \frac{-2\sqrt{2}}{2} = -\sqrt{2} .$$

4. Find the equation of the tangent line to $C : x = t^3 - 3t$, $y = t^2 - 5t - 1$ at $t = 2$.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2t - 5}{3t^2 - 3}$$

$$\text{The slope of the tangent line is } \frac{dy}{dx}\bigg|_{t=2} = \frac{2(2) - 5}{3(4) - 3} = \frac{-1}{9}$$

$$\text{At } t = 2 : x = (2)^3 - 3(2) = 8 - 6 = 2 \text{ and } y = (2)^2 - 5(2) - 1 = -7$$

The tangent line to C at $t = 2$ passes through the point $(2, -7)$ and its slope is $-\frac{1}{9}$, therefore its equation is $\frac{y + 7}{x - 2} = -\frac{1}{9}$

5. Find the points on $C : x = e^t$, $y = e^{-t}$ at which the slope of the tangent line to C equals $-e^{-2}$

$$m = \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-e^{-t}}{e^t} = -e^{-2t}$$

$$m = -e^{-2} \Rightarrow -e^{-2t} = -e^{-2} \Rightarrow t = 1 .$$

$$\text{At } t = 1 : x = e^1 = e \text{ and } y = e^{-1} = \frac{1}{e} .$$

Hence, the point at which the slope of the tangent line to C equals $-e^{-2}$ is $\left(e, \frac{1}{e}\right)$.

6. Find the points on $C : x = 4 + 4 \cos t$, $y = -1 + \sin t$; $0 \leq t \leq 2\pi$ at which the tangent line is : (a) Vertical , (b) Horizontal .

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cos t}{-4 \sin t}$$

- (a) The tangent line is vertical if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$$\frac{dx}{dt} = 0 \Rightarrow -4 \sin t = 0 \Rightarrow t = 0, t = \pi$$

Note that $0, \pi \in [0, 2\pi]$ and $\frac{dy}{dt} \neq 0$ at $t = 0$ or $t = \pi$.

At $t = 0 : x = 4 + 4(1) = 8$ and $y = -1 + 0 = -1$.

At $t = \pi : x = 4 + 4(-1) = 0$ and $y = -1 + 0 = -1$.

Hence, The tangent line to C is vertical at the points $(8, -1)$ and $(0, -1)$.

- (b) The tangent line is horizontal if $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

$$\frac{dy}{dt} = 0 \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2}, t = \frac{3\pi}{2}$$

Note that $\frac{\pi}{2}, \frac{3\pi}{2} \in [0, 2\pi]$ and $\frac{dx}{dt} \neq 0$ at $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$.

At $t = \frac{\pi}{2} : x = 4 + 4(0) = 4$ and $y = -1 + 1 = 0$.

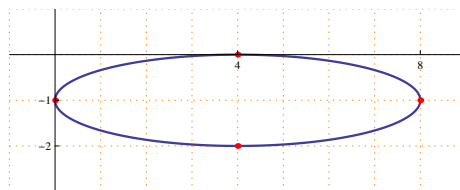
At $t = \frac{3\pi}{2} : x = 4 + 4(0) = 4$ and $y = -1 + (-1) = -2$.

Hence, The tangent line to C is horizontal at the points $(4, 0)$ and $(4, -2)$.

Note : $C : x = 4 + 4 \cos t$, $y = -1 + \sin t$; $0 \leq t \leq 2\pi$ represents the ellipse $\frac{(x-4)^2}{16} + \frac{(y+1)^2}{1} = 1$, with center = $(4, -1)$, the endpoints of the major axis are $(0, -1)$ and $(8, -1)$, the endpoints of the minor axis are $(4, 0)$ and $(4, -2)$.

Clearly, there are two vertical tangent lines to C , one passes through $(-1, 0)$ and the other passes through $(8, -1)$.

Also, there are two horizontal tangent lines to C , one passes through $(4, 0)$ and the other passes through $(4, -2)$



Exercises :

1. If $C : x = t, y = t^2$, find the slope of the tangent line to C at $t = 1$.
2. The point at which the curve $C : x = 3 \cos t, y = 3 \sin t; 0 \leq t \leq \pi$ has horizontal tangent line is

(a) $(0, 3)$ (b) $(3, 3)$ (c) $(3, 0)$ (d) None of these

(Hint : the parametric curve is the upper half of the circle with center = $(0, 0)$ and radius = 3).

ARC LENGTH OF A PARAMETRIC CURVE

If $C : x = x(t) , y = y(t) ; a \leq t \leq b$ is a differentiable parametric curve ,then its arc length equals $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt .$

Examples : Find the arc length of the following parametric curves :

1. $C : x = \frac{1}{3}t^3 + 1 , y = \frac{1}{2}t^2 + 2 ; 0 \leq t \leq 2$

$$\frac{dx}{dt} = t^2 \text{ and } \frac{dy}{dt} = t$$

$$L = \int_0^2 \sqrt{(t^2)^2 + (t)^2} dt = \int_0^2 \sqrt{t^4 + t^2} dt = \int_0^2 \sqrt{t^2(t^2 + 1)} dt$$

$$L = \int_0^2 |t|\sqrt{t^2 + 1} dt = \frac{1}{2} \int_0^2 (t^2 + 1)^{\frac{1}{2}}(2t) dt$$

$$L = \frac{1}{2} \left[\frac{2}{3}(t^2 + 1)^{\frac{3}{2}} \right]_0^2 = \frac{1}{3} (5\sqrt{5} - 1) .$$

2. $C : x = \sin t , y = \cos t ; 0 \leq t \leq \frac{\pi}{2}$

$$\frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = -\sin t$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{(\cos t)^2 + (-\sin t)^2} dt = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t + \sin^2 t} dt$$

$$L = \int_0^{\frac{\pi}{2}} dt = [t]_0^{\frac{\pi}{2}} = \frac{\pi}{2} .$$

Note : The parametric curve represents the first quarter of the unit circle, therefore its arc length equals $\frac{2\pi}{4} = \frac{\pi}{2} .$

3. $C : x = e^t \cos t , y = e^t \sin t ; 0 \leq t \leq \pi$

$$\frac{dx}{dt} = e^t \cos t - e^t \sin t = e^t(\cos t - \sin t)$$

$$\frac{dy}{dt} = e^t \sin t + e^t \cos t = e^t(\sin t + \cos t)$$

$$L = \int_0^{\pi} \sqrt{[e^t(\cos t - \sin t)]^2 + [e^t(\cos t + \sin t)]^2} dt$$

$$L = \int_0^{\pi} \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2} dt$$

$$L = \int_0^{\pi} \sqrt{e^{2t}(\cos^2 t - 2 \cos t \sin t + \sin^2 t + \cos^2 t + 2 \cos t \sin t + \sin^2 t)} dt$$

$$L = \int_0^\pi \sqrt{2e^{2t}} dt = \int_0^\pi \sqrt{2}|e^t| dt = \sqrt{2} \int_0^\pi e^t dt$$

$$L = \sqrt{2} [e^t]_0^\pi = \sqrt{2}(e^\pi - 1) .$$

SURFACE AREA GENERATED BY REVOLVING A PARAMETRIC CURVE

If $C : x = x(t)$, $y = y(t)$; $a \leq t \leq b$ is a differentiable parametric curve ,then the surface area generated by revolving C around the x-axis is

$$SA = 2\pi \int_a^b |y(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt .$$

The surface area generated by revolving C around the y-axis is

$$SA = 2\pi \int_a^b |x(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt .$$

Examples : Find the surface area generated by revolving the following parametric curves :

1. $C : x = t$, $y = \frac{1}{3}t^3 + \frac{1}{4}t^{-1}$; $1 \leq t \leq 2$, around the x-axis .

$$\frac{dx}{dt} = 1$$

$$\frac{dy}{dt} = t^2 - \frac{t^{-2}}{4}$$

$$SA = 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \sqrt{(1)^2 + \left(t^2 - \frac{t^{-2}}{4}\right)^2} dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \sqrt{1 + \left(t^4 - \frac{1}{2} + \frac{t^{-4}}{16}\right)} dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \sqrt{t^4 + \frac{1}{2} + \frac{t^{-4}}{16}} dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \sqrt{\left(t^2 + \frac{t^{-2}}{4}\right)^2} dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \left|t^2 + \frac{t^{-2}}{4}\right| dt$$

$$= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{t^{-1}}{4}\right) \left(t^2 + \frac{t^{-2}}{4}\right) dt$$

$$= 2\pi \int_1^2 \left(\frac{t^5}{3} + \frac{t}{2} + \frac{t^{-3}}{16}\right) dt$$

$$SA = 2\pi \left[\frac{t^6}{18} + \frac{t^2}{4} - \frac{t^{-2}}{32}\right]_1^2 = \frac{547\pi}{64}$$

2. $C : x = 4t^{\frac{1}{2}}$, $y = \frac{1}{2}t^2 + t^{-1}$; $1 \leq t \leq 4$, around the y-axis .

$$\frac{dx}{dt} = 2t^{-\frac{1}{2}}$$

$$\begin{aligned}
\frac{dy}{dt} &= t - t^{-2} \\
SA &= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \sqrt{\left(2t^{-\frac{1}{2}}\right)^2 + (t - t^{-2})^2} dt \\
&= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \sqrt{4t^{-1} + (t^2 - 2t^{-1} + t^{-4})} dt \\
&= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \sqrt{t^2 + 2t^{-1} + t^{-4}} dt \\
&= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) \sqrt{(t + t^{-2})^2} dt \\
&= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) |t + t^{-2}| dt \\
&= 2\pi \int_1^4 \left(4t^{\frac{1}{2}}\right) (t + t^{-2}) dt \\
&= 8\pi \int_1^4 \left(t^{\frac{3}{2}} + t^{-\frac{3}{2}}\right) dt \\
SA &= 8\pi \left[\frac{2}{5}t^{\frac{5}{2}} - 2t^{-\frac{1}{2}} \right]_1^4 = \frac{536\pi}{5}
\end{aligned}$$

Exercises : Find the surface area generated by revolving the following parametric curves :

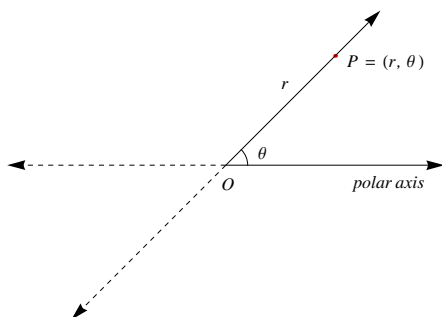
1. $C : x = 3t, y = 4t, 0 \leq t \leq 2$, around the x-axis .
2. $C : x = t, y = 2t, 0 \leq t \leq 4$, around the y-axis .

POLAR COORDINATES

In the rectangular coordinates system the ordered pair (a, b) represents a point, where "a" is the x-coordinate and "b" is the y-coordinate.

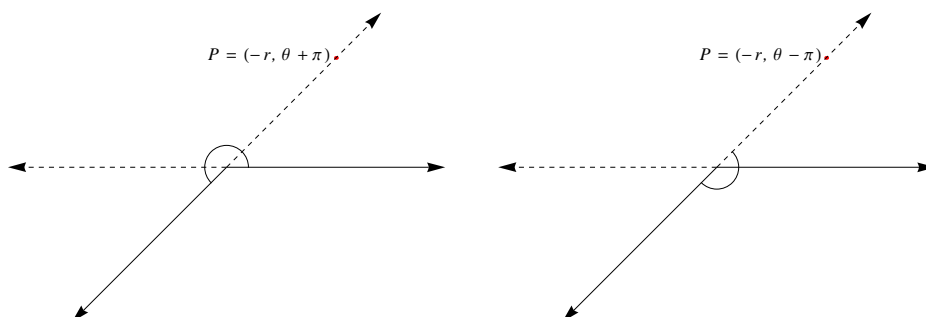
The polar coordinates system can be used also to represent points in the plane. The **pole** in the polar coordinates system is the origin in the rectangular coordinates system, and the **polar axis** is the directed half-line (the non-negative part of the x-axis).

If P is any point in the plane different from the origin, then its polar coordinates consist of two components r and θ , where r is the distance between P and the pole O , and θ is the measure of the angle determined by the polar axis and OP .



Note : The polar coordinates of a point is not unique, if $P = (r, \theta)$ then other representations are :

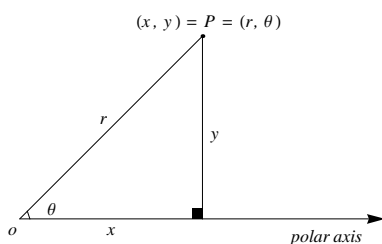
1. $P = (r, \theta + 2n\pi)$, where $n \in \mathbb{Z}$.
2. $P = (-r, \theta + \pi)$.
3. $P = (-r, \theta + \pi + 2n\pi)$, where $n \in \mathbb{Z}$.
4. $P = (-r, \theta - \pi)$
5. $P = (-r, \theta - \pi + 2n\pi)$, where $n \in \mathbb{Z}$.



Relationship between the polar and the rectangular coordinates

The polar coordinates (r, θ) and the rectangular coordinates (x, y) of a point P are related as follows :

1. $x = r \cos \theta$ and $y = r \sin \theta$.
2. $r^2 = x^2 + y^2$ and $\tan \theta = \frac{y}{x}$.



Examples :

1. If $(r, \theta) = \left(2, \frac{\pi}{2}\right)$ then its other polar coordinates is

a) $\left(-2, \frac{\pi}{2}\right)$ b) $\left(-2, \frac{3\pi}{2}\right)$ c) $\left(2, \frac{3\pi}{2}\right)$ d) $(2, \pi)$

The answer : $(r, \theta) = \left(2, \frac{\pi}{2}\right) = \left(-2, \frac{\pi}{2} + \pi\right) = \left(-2, \frac{3\pi}{2}\right)$

The right answer is (b) .

2. If $(r, \theta) = \left(-3, \frac{5\pi}{4}\right)$ then its other polar coordinates is

a) $\left(-3, \frac{3\pi}{4}\right)$ b) $\left(3, \frac{7\pi}{4}\right)$ c) $\left(3, \frac{\pi}{4}\right)$ d) $\left(-3, \frac{\pi}{4}\right)$

The answer : $(r, \theta) = \left(-3, \frac{5\pi}{4}\right) = \left(-(-3), \frac{5\pi}{4} - \pi\right) = \left(3, \frac{\pi}{4}\right)$

The right answer is (c) .

3. If $(r, \theta) = (-5, \pi)$ then find its rectangular coordinates (x, y) .

$$x = -5 \cos(\pi) = -5 (-1) = 5 \text{ and } y = -5 \sin(\pi) = -5 (0) = 0$$

$$(x, y) = (5, 0) .$$

4. If $(x, y) = (2\sqrt{3}, -2)$ then find its polar coordinates (r, θ) .

$$r^2 = (2\sqrt{3})^2 + (-2)^2 = 12 + 4 = 16 \Rightarrow r = 4$$

$$\tan \theta = \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}} \Rightarrow \theta = -\frac{\pi}{6}, \theta = \frac{11\pi}{6}$$

$$(r, \theta) = \left(4, -\frac{\pi}{6}\right) = \left(4, \frac{11\pi}{6}\right)$$

Exercises :

1. If $(r, \theta) = \left(2, \frac{\pi}{2}\right)$ then find its rectangular coordinates (x, y) .

Answer : $(x, y) = (0, 2)$.

2. If $(x, y) = (\sqrt{2}, \sqrt{2})$ then find its polar coordinates (r, θ) .

Answer : $\left(2, \frac{\pi}{4}\right)$.

POLAR CURVES

A polar curve is an equation in r and θ of the form $r = r(\theta)$.

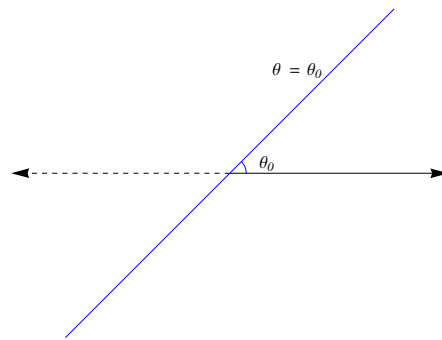
First - Straight Lines :

(1) Lines passing through the pole :

Any straight line passing through the pole has the form $\theta = \theta_0$, where θ_0 is the angle between the straight line and the polar axis.

$$\theta = \theta_0 \Rightarrow \tan(\theta) = \tan(\theta_0) \Rightarrow \frac{y}{x} = \tan(\theta_0) \Rightarrow y = \tan(\theta_0) x$$

The straight line $\theta = \theta_0$ is passing through the pole with a slope equals to $\tan(\theta_0)$.

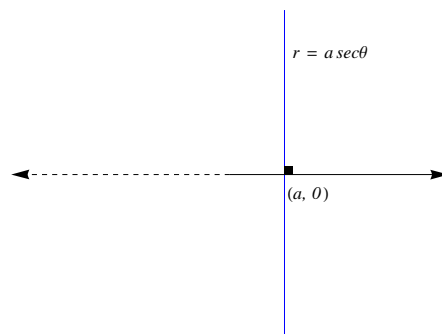


(2) Lines perpendicular to the polar axis :

Any straight line perpendicular to the polar axis has the form $r = a \sec \theta$, where $a \in \mathbb{R}^*$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$r = a \sec \theta \Rightarrow r = \frac{a}{\cos \theta} \Rightarrow r \cos \theta = a \Rightarrow x = a.$$

The straight line $r = a \sec \theta$ is perpendicular to the polar axis at the point $(r, \theta) = (a, 0)$.

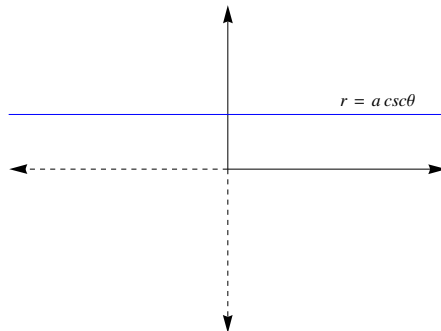


(3) Lines parallel to the polar axis :

Any straight line parallel to the polar axis has the form $r = a \csc \theta$, where $a \in \mathbb{R}^*$ and $\theta \in (0, \pi)$..

$$r = a \csc \theta \Rightarrow r = \frac{a}{\sin \theta} \Rightarrow r \sin \theta = a \Rightarrow y = a .$$

The straight line $r = a \sec \theta$ is parallel to the polar axis and passing through the point $(r, \theta) = \left(a, \frac{\pi}{2}\right)$.



Examples :

1. $\theta = \frac{\pi}{4}$ is a straight line passing through the pole with a slope equals to $\tan\left(\frac{\pi}{4}\right) = 1$. Therefore its equation in xy - form is $y = x$.
2. $r = 3 \sec \theta$ is a straight line perpendicular to the polar axis and passing through the point $(r, \theta) = (3, 0)$. Therefore its equation in xy - form is $x = 3$.
3. $r = -2 \csc \theta$ is a straight line parallel to the polar axis and passing through the point $(r, \theta) = \left(-2, \frac{\pi}{2}\right)$. Therefore its equation in the xy - form is $y = -2$.

Second - Circles :

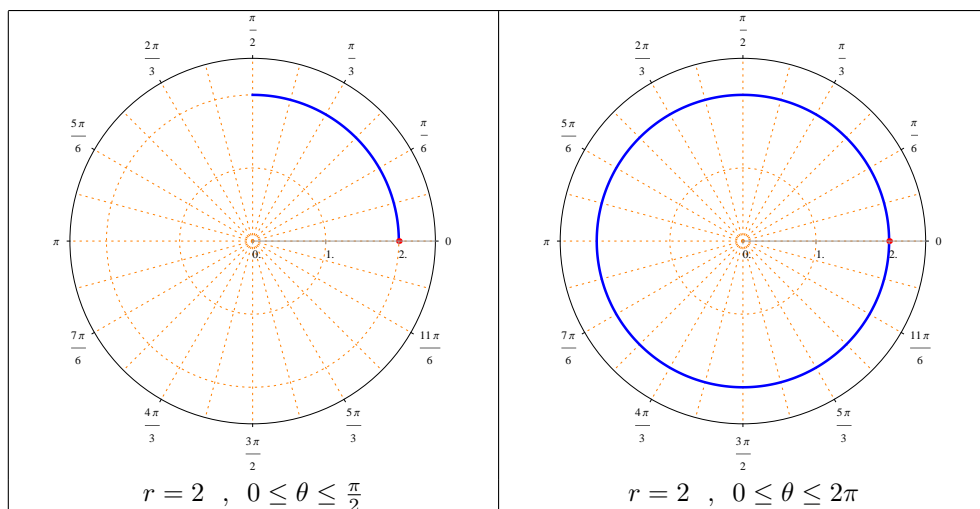
(1) **Circles of the form $r = a$** , where $a \in \mathbb{R}^*$

$$r = a \Rightarrow r^2 = a^2 \Rightarrow x^2 + y^2 = a^2$$

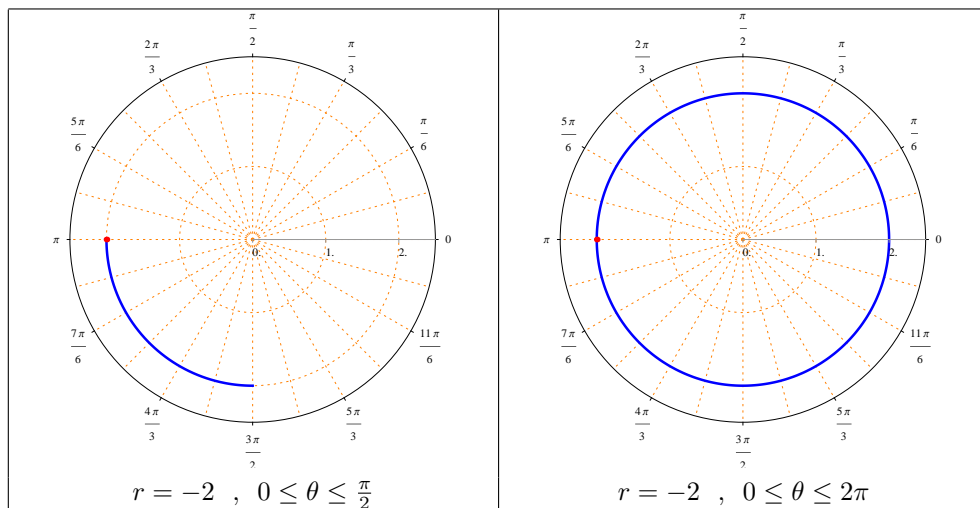
Therefore, $r = a$ represents a circle with center = $(0, 0)$ and radius equals $|a|$.

Example :

1. $r = 2$ represents a circle with center = $(0, 0)$ and radius equals to 2 .



2. $r = -2$ represents a circle with center = $(0, 0)$ and radius equals to 2 .



(2) Circles of the form $r = a \sin \theta$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq \pi$

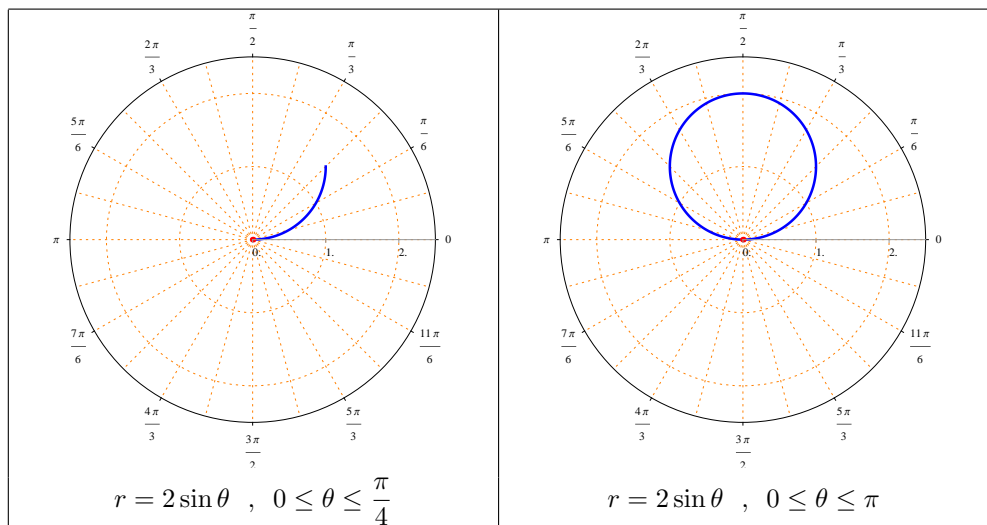
$$r = a \sin \theta \Rightarrow r^2 = a r \sin \theta \Rightarrow x^2 + y^2 = ay \Rightarrow x^2 + y^2 - ay = 0$$

$$\Rightarrow x^2 + \left(y^2 - ay + \frac{a^2}{4}\right) = \frac{a^2}{4} \Rightarrow x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$$

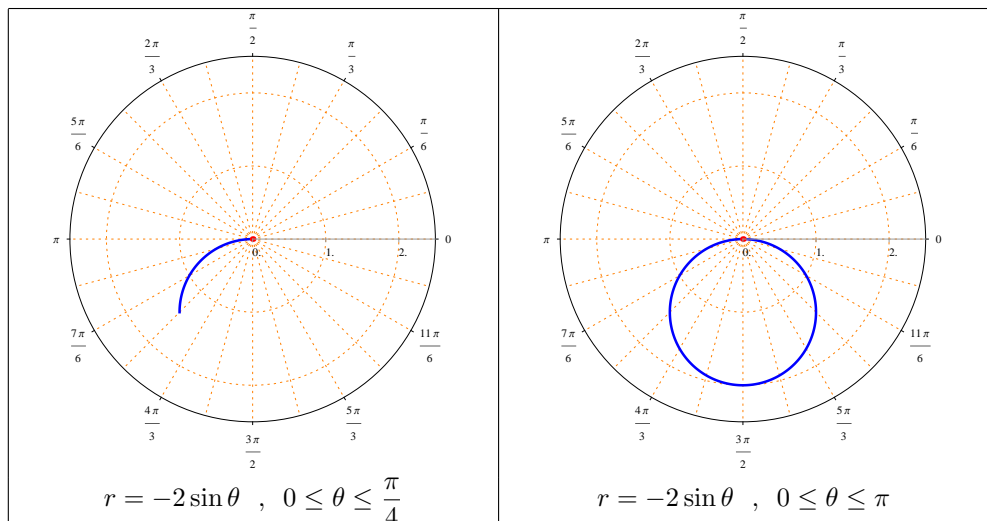
Therefore, $r = a \sin \theta$ represents a circle with center = $\left(0, \frac{a}{2}\right)$ and radius equals to $\frac{|a|}{2}$.

Examples :

- $r = 2 \sin \theta$ represents a circle with center = $(0, 1)$ and radius equals to 1



- $r = -2 \sin \theta$ represents a circle with center = $(0, -1)$ and radius equals to 1



(3) Circles of the form $r = a \cos \theta$, where $a \in \mathbb{R}^*$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

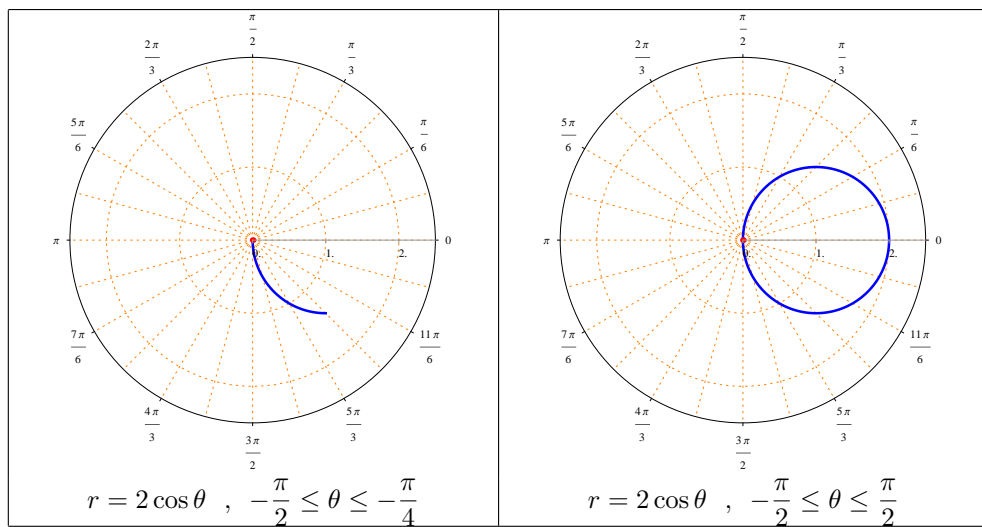
$$r = a \cos \theta \Rightarrow r^2 = a r \cos \theta \Rightarrow x^2 + y^2 = ax \Rightarrow x^2 - ax + y^2 = 0$$

$$\Rightarrow \left(x^2 - ax + \frac{a^2}{4}\right) + y^2 = \frac{a^2}{4} \Rightarrow \left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$$

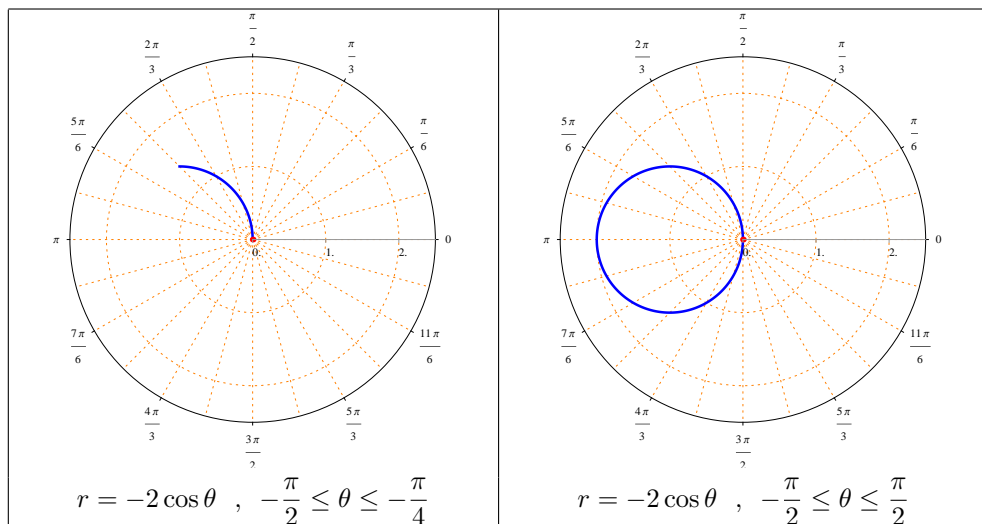
Therefore, $r = a \cos \theta$ represents a circle with center $= \left(\frac{a}{2}, 0\right)$ and radius equals to $\frac{|a|}{2}$.

Examples :

1. $r = 2 \cos \theta$ represents a circle with center $= (1, 0)$ and radius equals to 1



2. $r = -2 \cos \theta$ represents a circle with center $= (-1, 0)$ and radius equals to 1



Third - Limaçon curves :

The general form of a Limaçon curve is

$$r(\theta) = a + b \sin \theta \text{ or } r(\theta) = a + b \cos \theta, \text{ where } a, b \in \mathbb{R}^* \text{ and } 0 \leq \theta \leq 2\pi$$

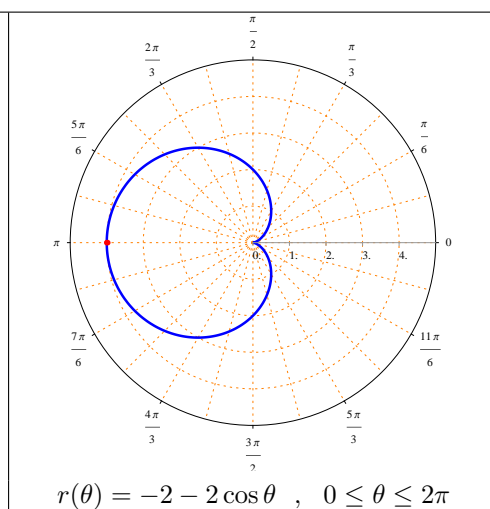
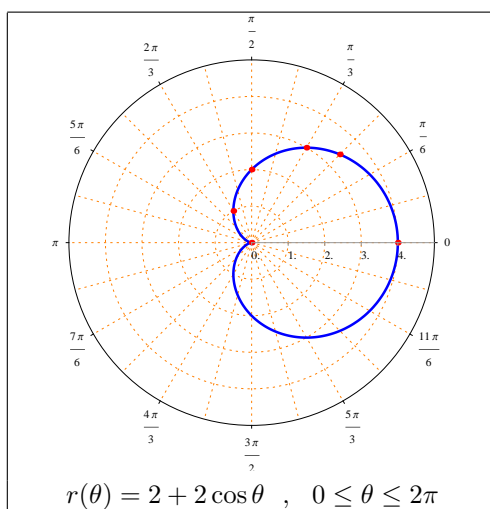
(1) Cardioid (Heart-shaped) :

It has the form $r(\theta) = a + a \sin \theta$ or $r(\theta) = a + a \cos \theta$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq 2\pi$

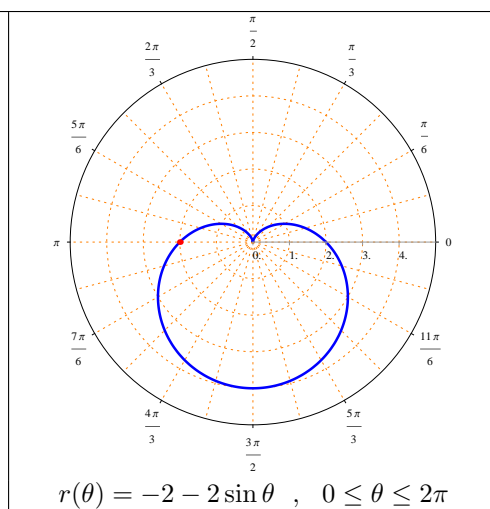
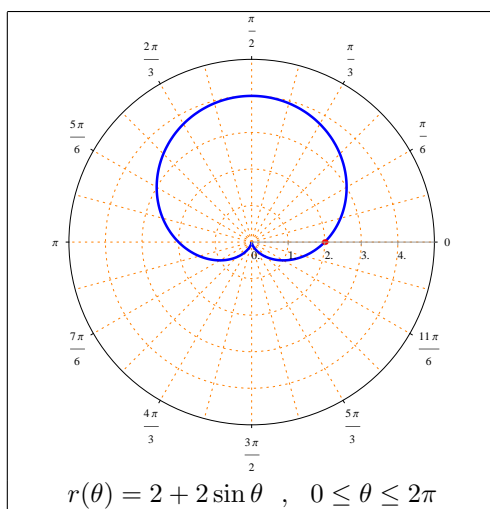
Examples :

1. $r(\theta) = 2 + 2 \cos \theta$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	$2 + \sqrt{2}$	3	2	1	0



2. $r(\theta) = 2 + 2 \sin \theta$ and $r(\theta) = -2 - 2 \sin \theta$



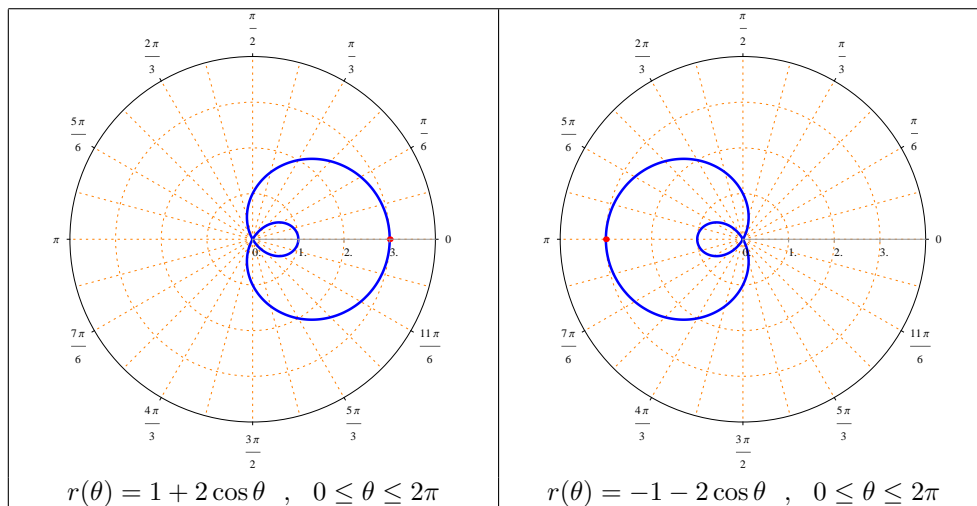
(2) Limaçon with inner loop :

It has the form $r(\theta) = a + b \sin \theta$ or $r(\theta) = a + b \cos \theta$, where $a, b \in \mathbb{R}^*$, $|a| < |b|$ and $0 \leq \theta \leq 2\pi$

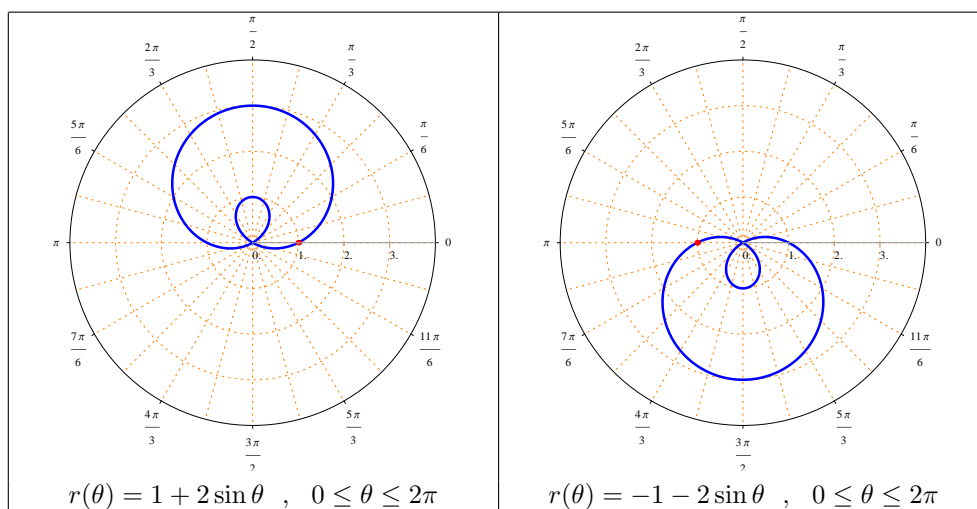
Note : Note that $|a| < |b|$ in this case .

Examples :

1. $r(\theta) = 1 + 2 \cos \theta$ and $r(\theta) = -1 - 2 \cos \theta$



2. $r(\theta) = 1 + 2 \sin \theta$ and $r(\theta) = -1 - 2 \sin \theta$



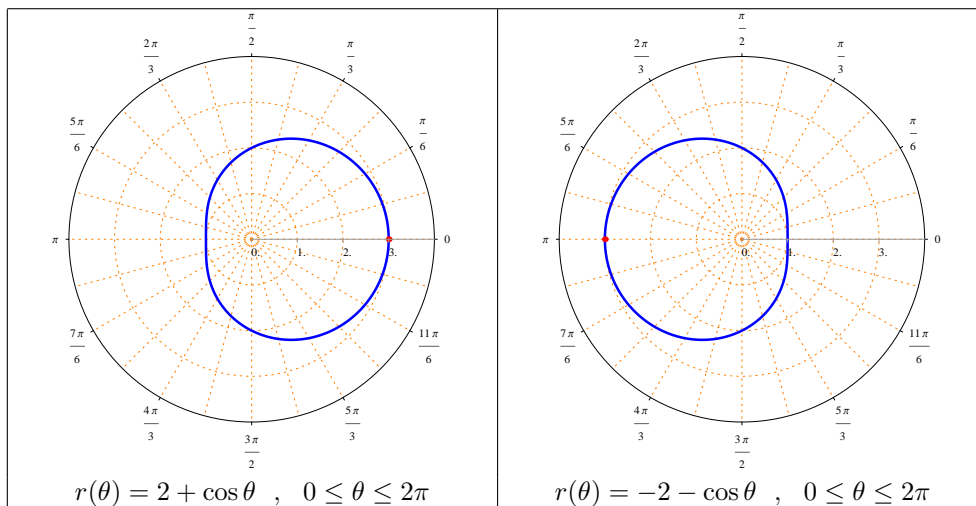
(3) Dimpled Limaçon :

It has the form $r(\theta) = a + b \sin \theta$ or $r(\theta) = a + b \cos \theta$, where $a, b \in \mathbb{R}^*$, $|a| > |b|$ and $0 \leq \theta \leq 2\pi$

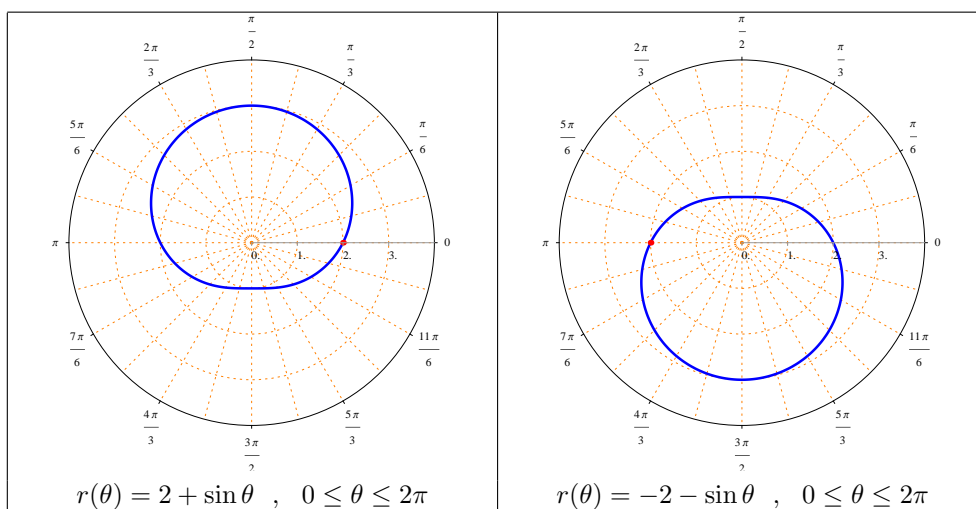
Note : Note that $|a| > |b|$ in this case .

Examples :

1. $r(\theta) = 2 + \cos \theta$ and $r(\theta) = -2 - \cos \theta$



2. $r(\theta) = 2 + \sin \theta$ and $r(\theta) = -2 - \sin \theta$



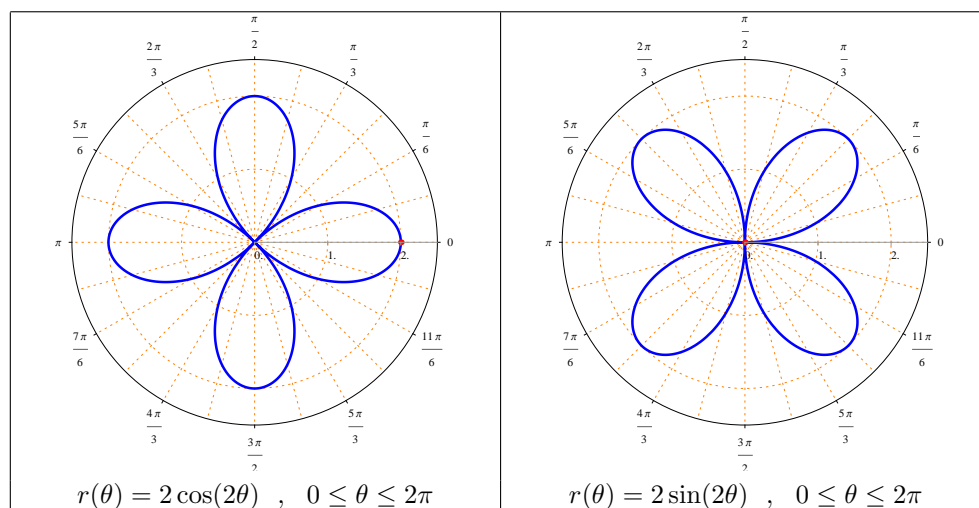
Fourth - Rose curves :

It has the form $r(\theta) = a \cos(n\theta)$ or $r(\theta) = a \sin(n\theta)$, where $a \in \mathbb{R}^*$, $n \in \mathbb{N}$ and $n \geq 2$

- n is even :** In this case the number of loops (or leaves) is $2n$.

Examples : $r(\theta) = 2 \cos(2\theta)$ or $r(\theta) = 2 \sin(2\theta)$, $0 \leq \theta \leq 2\pi$

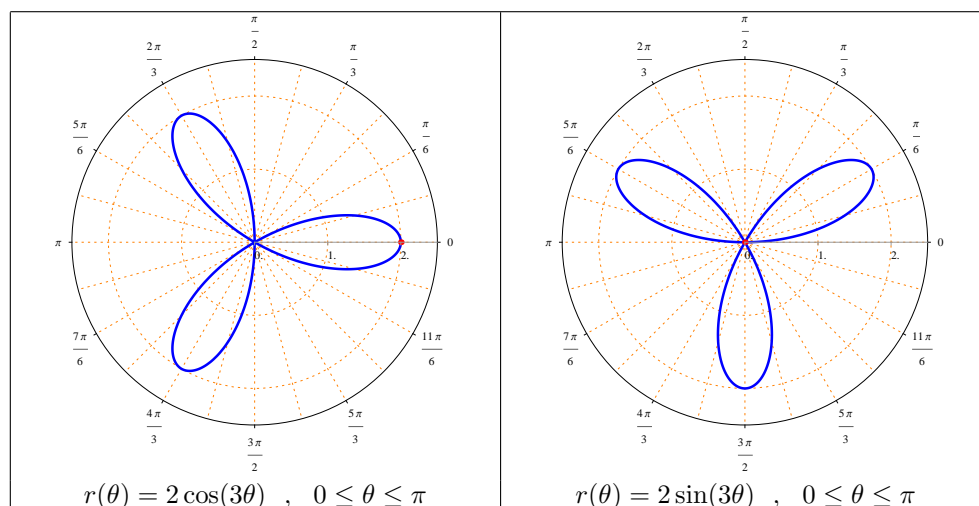
The number of loops (or leaves) equals 4 .



- n is odd :** In this case the number of loops (or leaves) is n .

Examples : $r(\theta) = 2 \cos(3\theta)$ or $r(\theta) = 2 \sin(3\theta)$, $0 \leq \theta \leq \pi$

The number of loops (or leaves) equals 3 .



Examples :

1. $r = \frac{2}{\cos \theta}$ represents
a) a straight line b) a circle c) a cardioid d) a rose curve

Answer : $r = \frac{2}{\cos \theta} \Rightarrow r \cos \theta = 2 \Rightarrow x = 2.$

Hence , $r = \frac{2}{\cos \theta}$ represents a straight line .

The right answer is (a) .

2. The polar equation $r = 2 \cos \theta - 2$ represents
a) a straight line b) a circle c) a cardioid d) a rose curve

$r = 2 \cos \theta - 2$ is a Limaçon curve with $a = b = 2$.

Therefore , $r = 2 \cos \theta - 2$ represents a cardioid .

The right answer is (c) .

3. The number of leaves in the rose curve $r = \sin 2\theta$ is
a) 6 b) 4 c) 2 d) None of these

Since $n = 2$ is an even number then the number of leaves in the rose curve $r = \sin 2\theta$ equals $2n = 2(2) = 4$

The right answer is (b)

4. Write the polar equation $r = 2 \cos \theta + 2 \sin \theta$ in terms of x and y (or cartesian equation) .

$$r = 2 \cos \theta + 2 \sin \theta \Rightarrow r^2 = 2 r \cos \theta + 2 r \sin \theta \Rightarrow x^2 + y^2 = 2x + 2y$$

$$\Rightarrow (x^2 - 2x + 1) + (y^2 - 2y + 1) = 2 \Rightarrow (x - 1)^2 + (y - 1)^2 = 2$$

It is a circle with center = $(1, 1)$ and radius equals $\sqrt{2}$

Test of symmetry

1. The graph of $r = r(\theta)$ is symmetric with respect to the polar axis if

$$r(\theta) = r(-\theta)$$

Examples : The circle $r = 4 \cos \theta$ and the cardioid $r = 2 + 2 \cos \theta$ are both symmetric with respect to the polar axis .

2. The graph of $r = r(\theta)$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$ if

(a) $r(\theta) = -r(-\theta)$

(b) $r(\theta) = r(\pi - \theta)$

Examples : The circle $r = 4 \sin \theta$ and the cardioid $r = 2 + 2 \sin \theta$ are both symmetric with respect to the line $\theta = \frac{\pi}{2}$.

3. The graph of $r = r(\theta)$ is symmetric with respect to the pole if

$$r(\theta) = r(\pi + \theta)$$

Example : The rose curve $r = \sin 2\theta$ is symmetric with respect to the pole .

SLOPE OF THE TANGENT LINE TO A POLAR CURVE

If $r = r(\theta)$ is a smooth polar curve, then the slope of the tangent line to $r = r(\theta)$ is $m = \frac{dy}{dx}$, where $x = r(\theta) \cos \theta$ and $y = r(\theta) \sin \theta$.

$$\text{More precisely, } m = \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

Notes :

1. The slope of the tangent line to $r = r(\theta)$ is horizontal if $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$
2. The slope of the tangent line to $r = r(\theta)$ is vertical if $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$

Example :

1. Find the points on the polar curve $r(\theta) = 2 \sin \theta$, $0 \leq \theta \leq \pi$ at which the tangent line to r is vertical.

The answer :

$$x = r(\theta) \cos \theta \Rightarrow x = 2 \sin \theta \cos \theta = \sin 2\theta \Rightarrow \frac{dx}{d\theta} = 2 \cos 2\theta$$

$$y = r(\theta) \sin \theta \Rightarrow y = 2 \sin^2 \theta \Rightarrow \frac{dy}{d\theta} = 4 \sin \theta \cos \theta$$

The tangent line to $r = r(\theta)$ is vertical if $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$

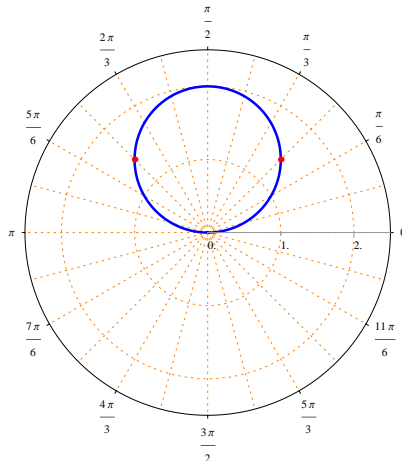
$$\frac{dx}{d\theta} = 0 \Rightarrow 2 \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}, 2\theta = \frac{3\pi}{2} \Rightarrow \theta = \frac{\pi}{4}, \theta = \frac{3\pi}{4}$$

Note that $\theta = \frac{\pi}{4}, \theta = \frac{3\pi}{4} \in [0, \pi]$ and $\frac{dy}{d\theta} \neq 0$ when $\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$.

$$\text{At } \theta = \frac{\pi}{4} : r\left(\frac{\pi}{4}\right) = 2 \sin\left(\frac{\pi}{4}\right) = 2 \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$\text{At } \theta = \frac{3\pi}{4} : r\left(\frac{3\pi}{4}\right) = 2 \sin\left(\frac{3\pi}{4}\right) = 2 \frac{1}{\sqrt{2}} = \sqrt{2}$$

The points on $r(\theta) = 2 \sin \theta$, $0 \leq \theta \leq \pi$ at which the tangent line to r is vertical are $\left(\sqrt{2}, \frac{\pi}{4}\right)$, $\left(\sqrt{2}, \frac{3\pi}{4}\right)$



2. Find the points on the polar curve $r(\theta) = 1 + \cos \theta$, $0 \leq \theta \leq 2\pi$ at which the tangent line to r is horizontal.

The answer :

$$x = r(\theta) \cos \theta \Rightarrow x = \cos \theta(1 + \cos \theta) = \cos \theta + \cos^2 \theta$$

$$y = r(\theta) \sin \theta \Rightarrow y = \sin \theta(1 + \cos \theta) = \sin \theta + \sin \theta \cos \theta = \sin \theta + \frac{1}{2} \sin 2\theta$$

$$\frac{dx}{d\theta} = -\sin \theta - 2 \cos \theta \sin \theta = -\sin \theta - \sin 2\theta$$

$$\frac{dy}{d\theta} = \cos \theta + \cos 2\theta$$

The tangent line to $r = r(\theta)$ is horizontal if $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$

$$\frac{dy}{d\theta} = 0 \Rightarrow \cos 2\theta + \cos \theta = 0 \Rightarrow 2 \cos^2 \theta - 1 + \cos \theta = 0$$

$$\Rightarrow (2 \cos \theta - 1)(\cos \theta + 1) = 0 \Rightarrow \cos \theta = -1 \text{ or } \cos \theta = \frac{1}{2}$$

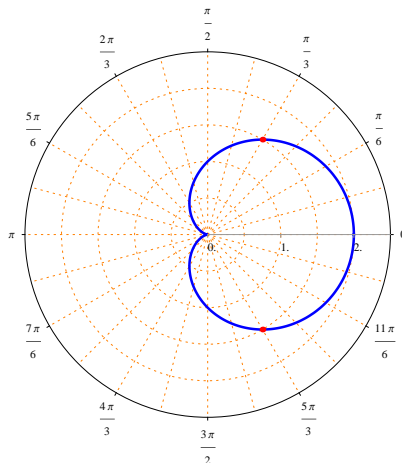
$$\Rightarrow \theta = \pi \text{ or } \theta = \frac{\pi}{3}, \theta = \frac{5\pi}{3}$$

Note that $\theta = \frac{\pi}{3}$, $\theta = \frac{5\pi}{3} \in [0, 2\pi]$ and $\frac{dx}{d\theta} \neq 0$ when $\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$, but $\frac{dx}{d\theta} = 0$ when $\theta = \pi$.

$$\text{At } \theta = \frac{\pi}{3} : r\left(\frac{\pi}{3}\right) = 1 + \cos\left(\frac{\pi}{3}\right) = 1 + \frac{1}{2} = \frac{3}{2}$$

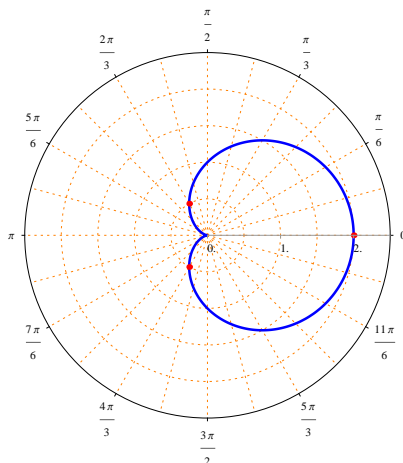
$$\text{At } \theta = \frac{5\pi}{3} : r\left(\frac{5\pi}{3}\right) = 1 + \cos\left(\frac{5\pi}{3}\right) = 1 + \frac{1}{2} = \frac{3}{2}$$

The points on $r(\theta) = 1 + \cos \theta$, $0 \leq \theta \leq 2\pi$ at which the tangent line to r is horizontal are $\left(\frac{3}{2}, \frac{\pi}{3}\right)$, $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$

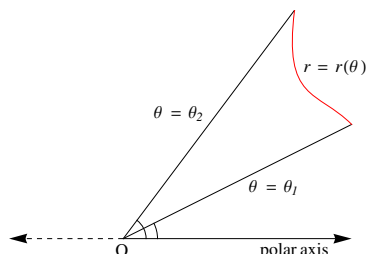


Exercise : Find the points on the polar curve $r(\theta) = 1 + \cos \theta$, $0 \leq \theta \leq 2\pi$ at which the tangent line to r is vertical .

The answer : $(2, 0)$, $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{1}{2}, \frac{4\pi}{3}\right)$.



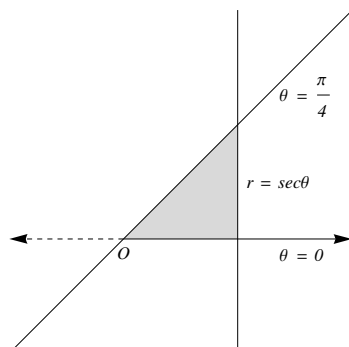
AREA INSIDE-BETWEEN POLAR CURVES



The area of the region bounded by the graphs of the polar curves $r = r(\theta)$, $\theta = \theta_1$ and $\theta = \theta_2$ is $A = \frac{1}{2} \int_{\theta_1}^{\theta_2} [r(\theta)]^2 d\theta$

Examples :

1. Find the area of the region bounded by the graph of the polar curves $r = \sec \theta$, $\theta = 0$ and $\theta = \frac{\pi}{4}$.

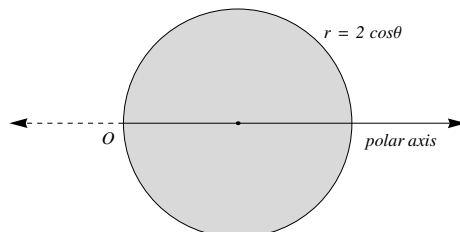


Note that $r = \sec \theta$ is a straight line perpendicular to the polar axis at the point $(r, \theta) = (1, 0)$, $\theta = 0$ is the polar axis and $\theta = \frac{\pi}{4}$ is a straight line passing the pole with a slope equals 1 (in fact it is the line $y = x$).

$$A = \frac{1}{2} \int_0^{\frac{\pi}{4}} (\sec \theta)^2 d\theta = \frac{1}{2} [\tan \theta]_0^{\frac{\pi}{4}} = \frac{1}{2} [1 - 0] = \frac{1}{2}$$

Note : In fact it is the area of the triangle of base equals 1 and height equals also 1.

2. Find the area inside the polar curve $r = 2 \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.



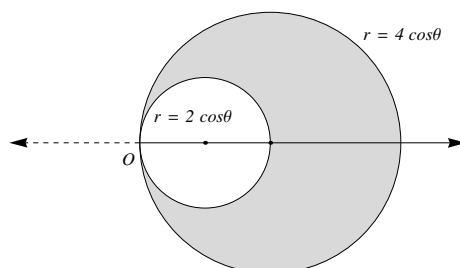
Note that $r = 2 \cos \theta$ is a circle with center = $(1, 0)$ and radius equals 1 .

$$A = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2 \cos \theta)^2 d\theta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \cos^2 \theta d\theta = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} [1 + \cos 2\theta] d\theta$$

$$A = \left[\theta + \frac{\sin 2\theta}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \left[\left(\frac{\pi}{2} + 0 \right) - \left(-\frac{\pi}{2} + 0 \right) \right] = \pi .$$

Note : In fact it is the area of a circle of radius equals 1 and in this case $A = \pi(1)^2 = \pi$.

3. Find the area inside the polar curve $r = 4 \cos \theta$ and outside the curve $r = 2 \cos \theta$.



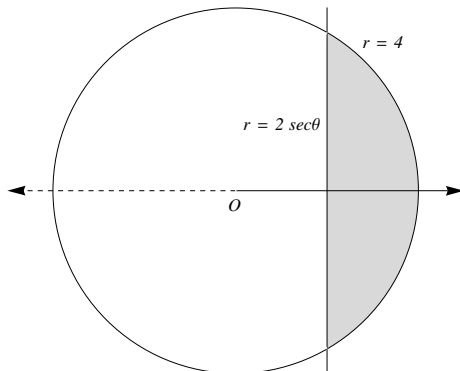
Note that $r = 4 \cos \theta$ is a circle with center = $(2, 0)$ and radius equals to 2 , also $r = 2 \cos \theta$ is another circle with center = $(1, 0)$ and radius equals 1 .

$$A = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4 \cos \theta)^2 d\theta - \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2 \cos \theta)^2 d\theta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 12 \cos^2 \theta d\theta$$

$$A = 6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} [1 + \cos 2\theta] d\theta = 3 \left[\theta + \frac{\sin 2\theta}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 3\pi$$

Note : In fact it is the difference between the area of a circle with radius 2 and the area of a circle of radius 1 , so the desired area is $A = \pi(2)^2 - \pi(1)^2 = 3\pi$.

4. Find the area inside $r = 4$ and to the right of $r = 2 \sec \theta$



Note that $r = 4$ is a circle with center = $(0, 0)$ and radius equals 4 ,
 $r = 2 \sec \theta$ is a straight line perpendicular to the polar axis (it is the line $x = 2$)

Angles of intersection between $r = 4$ and $r = 2 \sec \theta$:

$$2 \sec \theta = 4 \Rightarrow \sec \theta = 2 \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \theta = -\frac{\pi}{3}$$

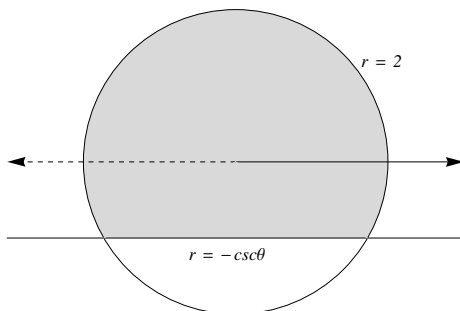
Since the desired area is symmetric with respect to the polar axis , then

$$A = 2 \left(\frac{1}{2} \int_0^{\frac{\pi}{3}} (4)^2 d\theta - \frac{1}{2} \int_0^{\frac{\pi}{3}} (2 \sec \theta)^2 d\theta \right)$$

$$A = 16 \int_0^{\frac{\pi}{3}} d\theta - 4 \int_0^{\frac{\pi}{3}} \sec^2 \theta d\theta$$

$$A = 16[\theta]_0^{\frac{\pi}{3}} - 4[\tan \theta]_0^{\frac{\pi}{3}} = 16 \left(\frac{\pi}{3} - 0 \right) - 4(\sqrt{3} - 0) = \frac{16\pi}{3} - 4\sqrt{3}$$

5. Find the area inside $r = 2$ and above $r = -\csc \theta$.



Note that $r = 2$ is a circle with center = $(0, 0)$ and radius equals 2 ,
 $r = -\csc \theta$ is a straight line parallel to the polar axis (it is the line $y = -1$)

Angles of intersection between $r = 2$ and $r = -\csc \theta$:

$$-\csc \theta = 2 \Rightarrow \csc \theta = -2 \Rightarrow \sin \theta = -\frac{1}{2} \Rightarrow \theta = -\frac{\pi}{6}, \theta = -\frac{5\pi}{6}$$

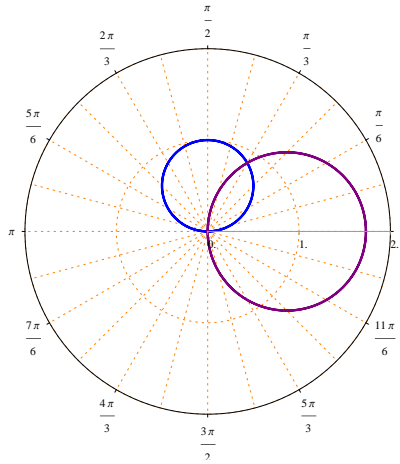
Since the desired area is symmetric with respect to the line $\theta = \frac{\pi}{2}$, then

$$A = 2 \left(\frac{1}{2} \int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} (-\csc \theta)^2 d\theta + \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (2)^2 d\theta \right)$$

$$A = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} \csc^2 \theta d\theta + 4 \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} d\theta$$

$$A = [-\cot \theta]_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} + 4[\theta]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} = \sqrt{3} + \frac{2\pi}{3}$$

6. Find the area of the common region between $r = \sqrt{3} \cos \theta$ and $r = \sin \theta$



Note that $r = \sqrt{3} \cos \theta$ is a circle with center $= \left(\frac{\sqrt{3}}{2}, 0\right)$ and radius equals $\frac{\sqrt{3}}{2}$, also $r = \sin \theta$ is a circle with center $= \left(0, \frac{1}{2}\right)$ and radius equals $\frac{1}{2}$.

Angle of intersection between $r = \sqrt{3} \cos \theta$ and $r = \sin \theta$

$$\sqrt{3} \cos \theta = \sin \theta \Rightarrow \tan \theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{3}$$

$$A = \frac{1}{2} \int_0^{\frac{\pi}{3}} (\sin \theta)^2 d\theta + \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sqrt{3} \cos \theta)^2 d\theta$$

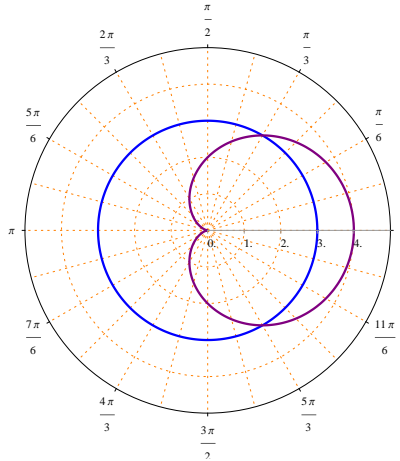
$$A = \frac{1}{2} \int_0^{\frac{\pi}{3}} \frac{1}{2} [1 - \cos 2\theta] d\theta + \frac{3}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2} [1 + \cos 2\theta] d\theta$$

$$A = \frac{1}{4} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{3}} + \frac{3}{4} \left[\theta + \frac{\sin 2\theta}{2} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$A = \frac{1}{4} \left(\frac{\pi}{3} - \frac{1\sqrt{3}}{2} \right) + \frac{3}{4} \left[\left(\frac{\pi}{2} + 0 \right) - \left(\frac{\pi}{3} + \frac{1\sqrt{3}}{2} \right) \right]$$

$$A = \frac{5\pi}{24} - \frac{\sqrt{3}}{4} .$$

7. Find the area inside $r = 3$ and outside $r = 2 + 2 \cos \theta$.



Note that $r = 3$ is a circle with center $= (0,0)$ and radius equals 3 ,
 $r = 2 + 2 \cos \theta$ is a cardioid .

Angles of intersection between $r = 3$ and $r = 2 + 2 \cos \theta$:

$$2 + 2 \cos \theta = 3 \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} , \theta = \frac{5\pi}{3} = -\frac{\pi}{3}$$

Since the desired area is symmetric with respect to the polar axis , then

$$A = 2 \left(\frac{1}{2} \int_{\frac{\pi}{3}}^{\pi} (3)^2 d\theta - \frac{1}{2} \int_{\frac{\pi}{3}}^{\pi} (2 + 2 \cos \theta)^2 d\theta \right)$$

$$A = \int_{\frac{\pi}{3}}^{\pi} [9 - (4 + 8 \cos \theta + 4 \cos^2 \theta)] d\theta$$

$$A = \int_{\frac{\pi}{3}}^{\pi} [5 - 8 \cos \theta - 2(1 + \cos 2\theta)] d\theta$$

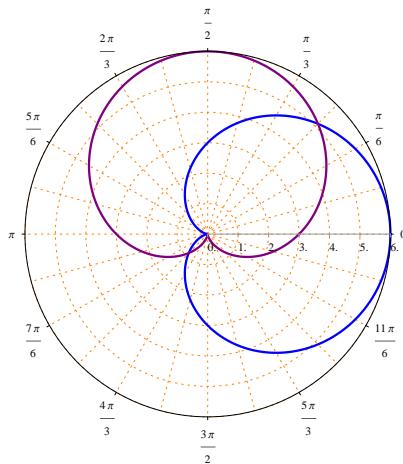
$$A = \int_{\frac{\pi}{3}}^{\pi} [3 - 8 \cos \theta - 2 \cos 2\theta] d\theta$$

$$A = [3\theta - 8 \sin \theta - \sin 2\theta]_{\frac{\pi}{3}}^{\pi}$$

$$A = \left[(3\pi - 0 - 0) - \left(\pi - 8 \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) \right]$$

$$A = 2\pi + \frac{9\sqrt{3}}{2}$$

8. Find the area inside $r = 3 + 3 \cos \theta$, outside $r = 3 + 3 \sin \theta$ and at the first quadrant.



Angles of intersection between $r = 3 + 3 \cos \theta$ and $r = 3 + 3 \sin \theta$:

$$3 + 3 \cos \theta = 3 + 3 \sin \theta \Rightarrow \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \theta = \frac{5\pi}{4}$$

$$A = \frac{1}{2} \int_0^{\frac{\pi}{4}} (3 + 3 \cos \theta)^2 d\theta - \frac{1}{2} \int_0^{\frac{\pi}{4}} (3 + 3 \sin \theta)^2 d\theta$$

$$A = \frac{1}{2} \int_0^{\frac{\pi}{4}} [(9 + 18 \cos \theta + 9 \cos^2 \theta) - (9 + 18 \sin \theta + 9 \sin^2 \theta)] d\theta$$

$$A = \frac{1}{2} \int_0^{\frac{\pi}{4}} [18 \cos \theta - 18 \sin \theta + 9 \cos^2 \theta - 9 \sin^2 \theta] d\theta$$

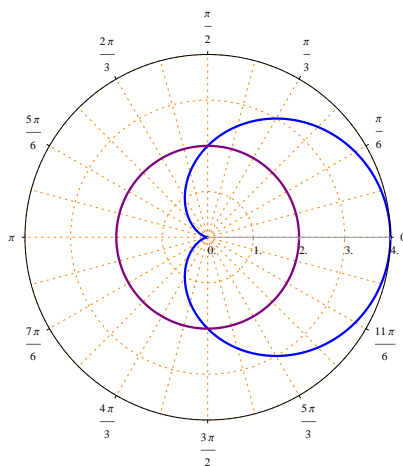
$$A = \frac{1}{2} \int_0^{\frac{\pi}{4}} \left[18 \cos \theta - 18 \sin \theta + \frac{9}{2}(1 + \cos 2\theta) - \frac{9}{2}(1 - \cos 2\theta) \right] d\theta$$

$$A = \frac{1}{2} \int_0^{\frac{\pi}{4}} [18 \cos \theta - 18 \sin \theta + 9 \cos 2\theta] d\theta$$

$$A = \frac{1}{2} \left[18 \sin \theta + 18 \cos \theta + \frac{9}{2} \sin 2\theta \right]_0^{\frac{\pi}{4}}$$

$$A = \frac{1}{2} \left[\left(\frac{18}{\sqrt{2}} + \frac{18}{\sqrt{2}} + \frac{9}{2} \right) - (0 + 18 + 0) \right] = \frac{18}{\sqrt{2}} - \frac{27}{4}$$

9. Find the area inside $r = 2 + 2 \cos \theta$ and outside $r = 2$.



Note that $r = 2$ is a circle with center = $(0, 0)$ and radius equals 2 ,
 $r = 2 + 2 \cos \theta$ is a cardioid .

Angles of intersection between $r = 2$ and $r = 2 + 2 \cos \theta$:

$$2 + 2 \cos \theta = 2 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}$$

Since the desired area is symmetric with respect to the polar axis , then

$$A = 2 \left(\frac{1}{2} \int_0^{\frac{\pi}{2}} (2 + 2 \cos \theta)^2 d\theta - \frac{1}{2} \int_0^{\frac{\pi}{2}} (2)^2 d\theta \right)$$

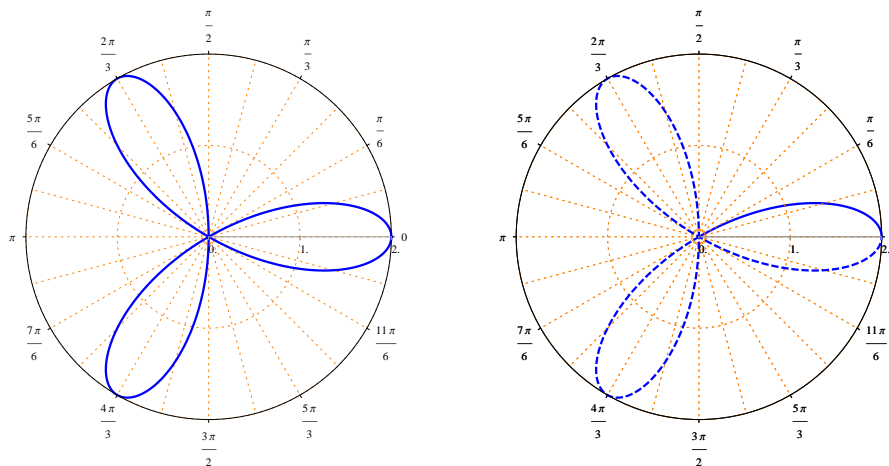
$$A = \int_0^{\frac{\pi}{2}} (4 + 8 \cos \theta + 4 \cos^2 \theta - 4) d\theta$$

$$A = \int_0^{\frac{\pi}{2}} (8 \cos \theta + 2(1 + \cos 2\theta)) d\theta$$

$$A = \int_0^{\frac{\pi}{2}} (2 + 8 \cos \theta + 2 \cos 2\theta) d\theta$$

$$A = [2\theta + 8 \sin \theta + \sin 2\theta]_0^{\frac{\pi}{2}} = \pi + 8$$

10. Find the area inside one leaf of the rose curve $r = 2 \cos 3\theta$.



The rose curve $r = 2 \cos 3\theta$, $0 \leq \theta \leq \pi$ starts at $(r, \theta) = (2, 0)$ and reaches the pole when $r = 0$

$$r = 0 \Rightarrow 2 \cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$$

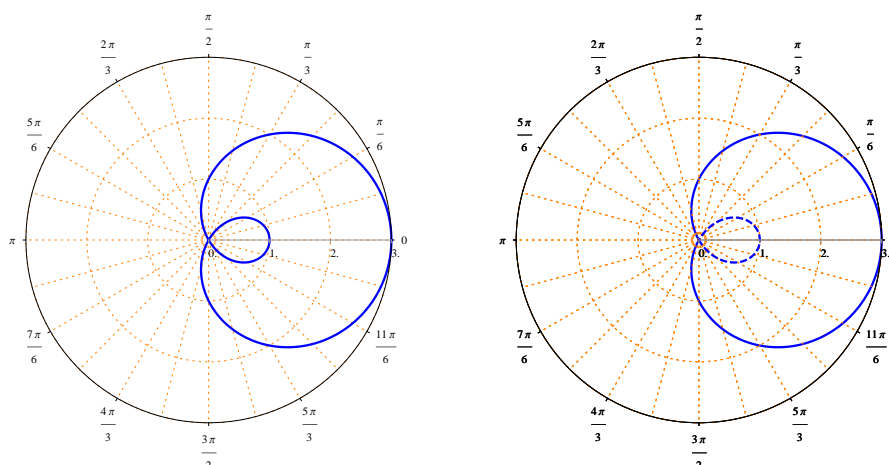
Since the desired area is symmetric with respect to the polar axis, then

$$A = 2 \left(\frac{1}{2} \int_0^{\frac{\pi}{6}} (2 \cos 3\theta)^2 d\theta \right) = 4 \int_0^{\frac{\pi}{6}} \cos^2 3\theta d\theta$$

$$A = 4 \int_0^{\frac{\pi}{6}} \frac{1}{2} (1 + \cos 6\theta) d\theta = 2 \int_0^{\frac{\pi}{6}} (1 + \cos 6\theta) d\theta$$

$$A = 2 \left[\theta + \frac{\sin 6\theta}{6} \right]_0^{\frac{\pi}{6}} = \frac{\pi}{3}$$

11. Find the area between the loops of the curve $r = 1 + 2 \cos \theta$



$$r = 0 \Rightarrow 1 + 2 \cos \theta = 0 \Rightarrow \cos \theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \theta = \frac{4\pi}{3}$$

The interior loop starts at $\theta = \frac{2\pi}{3}$ and ends at $\theta = \frac{4\pi}{3}$

$$A = \frac{1}{2} \int_0^{\frac{2\pi}{3}} (1 + 2 \cos \theta)^2 d\theta + \int_{\frac{4\pi}{3}}^{2\pi} (1 + 2 \cos \theta)^2 d\theta - \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 2 \cos \theta)^2 d\theta$$

Since the desired area is symmetric with respect to the polar axis, then

$$A = 2 \left(\frac{1}{2} \int_0^{\frac{2\pi}{3}} (1 + 2 \cos \theta)^2 d\theta - \frac{1}{2} \int_{\frac{2\pi}{3}}^{\pi} (1 + 2 \cos \theta)^2 d\theta \right)$$

$$A = \int_0^{\frac{2\pi}{3}} (1 + 4 \cos \theta + 4 \cos^2 \theta) d\theta - \int_{\frac{2\pi}{3}}^{\pi} (1 + 4 \cos \theta + 4 \cos^2 \theta) d\theta$$

$$A = \int_0^{\frac{2\pi}{3}} (3 + 4 \cos \theta + 2 \cos 2\theta) d\theta - \int_{\frac{2\pi}{3}}^{\pi} (3 + 4 \cos \theta + 2 \cos 2\theta) d\theta$$

$$A = [3\theta + 4 \sin \theta + \sin 2\theta]_0^{\frac{2\pi}{3}} - [3\theta + 4 \sin \theta + \sin 2\theta]_{\frac{2\pi}{3}}^{\pi}$$

$$A = \left[\left(2\pi + \frac{3\sqrt{3}}{2} \right) - 0 \right] - \left[3\pi - \left(2\pi + \frac{3\sqrt{3}}{2} \right) \right] = \pi + 3\sqrt{3}$$

Exercises :

1. Find the area inside $r = \cos \theta$ and outside the curve $r = 1 - \cos \theta$
2. Find the area of the common region between the curves $r = 2 \sin \theta$ and $r = 2 \cos \theta$
3. Find the area inside the curve $r = 1$ and outside the curve $r = 1 - \cos \theta$

ARC LENGTH OF A POLAR CURVE

The arc length of the polar curve $r = r(\theta)$ from θ_1 to θ_2 is

$$L = \int_{\theta_1}^{\theta_2} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Examples : Find the arc length of the following polar curves :

1. $r = 1 + \cos \theta$, $0 \leq \theta \leq 2\pi$

$$\frac{dr}{d\theta} = -\sin \theta$$

Since $r = 1 + \cos \theta$ is symmetric with respect to the polar axis then

$$L = 2 \int_0^{\pi} \sqrt{(1 + \cos \theta)^2 + (-\sin \theta)^2} d\theta$$

$$L = 2 \int_0^{\pi} \sqrt{(1 + 2 \cos \theta + \cos^2 \theta) + \sin^2 \theta} d\theta$$

$$L = 2 \int_0^{\pi} \sqrt{2 + 2 \cos \theta} d\theta$$

$$L = 2 \int_0^{\pi} \sqrt{2(1 + \cos \theta)} d\theta$$

Note that $\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 + \cos \theta) \Rightarrow 2(1 + \cos \theta) = 4 \cos^2\left(\frac{\theta}{2}\right)$

$$L = 2 \int_0^{\pi} \sqrt{4 \cos^2\left(\frac{\theta}{2}\right)} d\theta = 2 \int_0^{\pi} 2 \left| \cos\left(\frac{\theta}{2}\right) \right| d\theta$$

$$L = 4 \int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta = 8 \left[\sin\left(\frac{\theta}{2}\right) \right]_0^{\pi} = 8(1 - 0) = 8$$

2. $r = 2 \cos \theta$, $0 \leq \theta \leq 2\pi$

$$\frac{dr}{d\theta} = -2 \sin \theta$$

$$L = \int_0^{2\pi} \sqrt{(2 \cos \theta)^2 + (-2 \sin \theta)^2} d\theta$$

$$L = \int_0^{2\pi} \sqrt{4 \cos^2 \theta + 4 \sin^2 \theta} d\theta$$

$$L = \int_0^{2\pi} \sqrt{4} d\theta = \int_0^{2\pi} 2 d\theta = [2\theta]_0^{2\pi} = 4\pi$$

Note that $r = 2 \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is a circle with center $(1, 0)$ and radius equals 1 , therefore its circumference equals 2π , in this example $r = 2 \cos \theta$, $0 \leq \theta \leq 2\pi$ which means that the curve is doubled , hence the circumference is also doubled .

$$3. r = e^{-\theta}, 0 \leq \theta \leq \pi$$

$$\frac{dr}{d\theta} = -e^{-\theta}$$

$$L = \int_0^\pi \sqrt{(e^{-\theta})^2 + (-e^{-\theta})^2} d\theta$$

$$L = \int_0^\pi \sqrt{e^{-2\theta} + e^{-2\theta}} d\theta = \int_0^\pi \sqrt{2e^{-2\theta}} d\theta$$

$$L = \int_0^\pi \sqrt{2} |e^{-\theta}| d\theta = \sqrt{2} \int_0^\pi e^{-\theta} d\theta$$

$$L = \sqrt{2} [-e^{-\theta}]_0^\pi = \sqrt{2} [-e^{-\pi} + e^0] = \sqrt{2} (1 - e^{-\pi})$$

SURFACE AREA GENERATED BY REVOLVING A POLAR CURVE

The surface area generated by revolving the polar curve $r = r(\theta)$, $\theta_1 \leq \theta \leq \theta_2$ around the polar axis is

$$SA = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta) \sin \theta| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

The surface area generated by revolving the polar curve $r = r(\theta)$, $\theta_1 \leq \theta \leq \theta_2$ around the line $\theta = \frac{\pi}{2}$ is

$$SA = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta) \cos \theta| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Examples : Find the surface area generated by revolving the following polar curves :

1. $r = e^{\frac{\theta}{2}}$, $0 \leq \theta \leq \pi$, around the polar axis .

$$\frac{dr}{d\theta} = \frac{1}{2}e^{\frac{\theta}{2}}$$

$$SA = 2\pi \int_0^{\pi} \left|e^{\frac{\theta}{2}} \sin \theta\right| \sqrt{\left(e^{\frac{\theta}{2}}\right)^2 + \left(\frac{1}{2}e^{\frac{\theta}{2}}\right)^2} d\theta$$

$$SA = 2\pi \int_0^{\pi} e^{\frac{\theta}{2}} \sin \theta \sqrt{e^{\theta} + \frac{1}{4}e^{\theta}} d\theta = \int_0^{\pi} e^{\frac{\theta}{2}} \sin \theta \left|e^{\frac{\theta}{2}}\right| \sqrt{1 + \frac{1}{4}} d\theta$$

$$SA = 2\pi \int_0^{\pi} e^{\frac{\theta}{2}} \sin \theta e^{\frac{\theta}{2}} \sqrt{\frac{5}{4}} d\theta = 2\pi \frac{\sqrt{5}}{2} \int_0^{\pi} e^{\theta} \sin \theta d\theta$$

Using integration by parts

$$SA = \sqrt{5}\pi \left[\frac{1}{2}e^{\theta}(\sin \theta - \cos \theta) \right]_0^{\pi} = \frac{\sqrt{5}\pi}{2} (e^{\pi} + 1)$$

2. $r = 2 + 2 \cos \theta$, $0 \leq \theta \leq \frac{\pi}{2}$, around the polar axis .

$$\frac{dr}{d\theta} = -2 \sin \theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |(2 + 2 \cos \theta) \sin \theta| \sqrt{(2 + 2 \cos \theta)^2 + (-2 \sin \theta)^2} d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} (2 + 2 \cos \theta) \sin \theta \sqrt{4 + 8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta} d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} (2 + 2 \cos \theta) \sin \theta \sqrt{8 + 8 \cos \theta} d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} (2 + 2 \cos \theta) \sin \theta \sqrt{4(2 + 2 \cos \theta)} d\theta$$

$$SA = 4\pi \int_0^{\frac{\pi}{2}} (2 + 2 \cos \theta) \sin \theta \sqrt{2 + 2 \cos \theta} \, d\theta$$

$$SA = 4\pi \int_0^{\frac{\pi}{2}} (2 + 2 \cos \theta)^{\frac{3}{2}} \sin \theta \, d\theta$$

$$SA = -2\pi \int_0^{\frac{\pi}{2}} (2 + 2 \cos \theta)^{\frac{3}{2}} (-2 \sin \theta) \, d\theta$$

$$SA = -2\pi \left[\frac{2}{5} (2 + 2 \cos \theta)^{\frac{5}{2}} \right]_0^{\frac{\pi}{2}} = -2\pi \frac{2}{5} [4\sqrt{2} - 32] = \frac{16\pi}{5} (8 - \sqrt{2})$$

3. $r = \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, around the line $\theta = \frac{\pi}{2}$

$$\frac{dr}{d\theta} = -\sin \theta$$

$$SA = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos \theta \cos \theta| \sqrt{(\cos \theta)^2 + (-\sin \theta)^2} \, d\theta$$

$$SA = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos^2 \theta| \sqrt{\cos^2 \theta + \sin^2 \theta} \, d\theta$$

$$SA = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta$$

$$SA = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos 2\theta) \, d\theta$$

$$SA = \pi \left[\theta + \frac{\sin 2\theta}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi \left[\left(\frac{\pi}{2} + 0 \right) - \left(-\frac{\pi}{2} + 0 \right) \right] = \pi^2$$

4. $r = 2 \sin \theta$, $0 \leq \theta \leq \frac{\pi}{2}$, around the line $\theta = \frac{\pi}{2}$

$$\frac{dr}{d\theta} = 2 \cos \theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |2 \sin \theta \cos \theta| \sqrt{(2 \sin \theta)^2 + (2 \cos \theta)^2} \, d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |\sin 2\theta| \sqrt{4 \sin^2 \theta + 4 \cos^2 \theta} \, d\theta$$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} \sin 2\theta \sqrt{4} \, d\theta$$

$$SA = 4\pi \left[-\frac{\cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} = 4\pi$$

Note : it is the surface area of a sphere of radius 1.