M 104 - GENERAL MATHEMATICS -2-

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Solution of the First Mid-Term Exam

First semester 1438-1439 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} -2 & 3 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Compute (if possible) : **A+B** and **BC**

Solution :

A+B is impossible.

$$\mathbf{BC} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2+0 & 0+(-2) \\ -4+0 & 0+0 \\ 2+0 & 0+6 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -4 & 0 \\ 2 & 6 \end{pmatrix}$$

Q.2 Compute The determinant $\begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & 0 \\ 2 & 0 & 1 \end{vmatrix}$

Solution (1) : Using Sarrus Method

$$\begin{matrix} 3 & 2 & 1 & 3 & 2 \\ 0 & 4 & 0 & 0 & 4 \\ 2 & 0 & 1 & 2 & 0 \end{matrix}$$

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & 0 \\ 2 & 0 & 1 \end{vmatrix} = (12+0+0) - (8+0+0) = 12 - 8 = 4$$

Solution (2) : By the definition (using second row)

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 4 \times \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 4(3-2) = 4$$

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Q.3 Solve by Gauss the linear system :

$$\begin{cases} x - 2y + z = 5 \\ \quad\; y + 3z = 5 \\ -x + 3y - z = -6 \end{cases}$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ -1 & 3 & -1 & -6 \end{array}\right) \xrightarrow{R_1+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{-R_2+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

$-3z = -6 \implies z = 2$

$y + 3z = 5 \implies y + 6 = 5 \implies y = -1$

$x - 2y + z = 5 \implies x - 2(-1) + 2 = 5 \implies x = 1$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Q.4 Find the standard equation of the parabola with focus $F(5,1)$ and vertex $V(6,1)$, then sketch it.

Solution :

From the position of the focus and the vertex the parabola opens to the left.

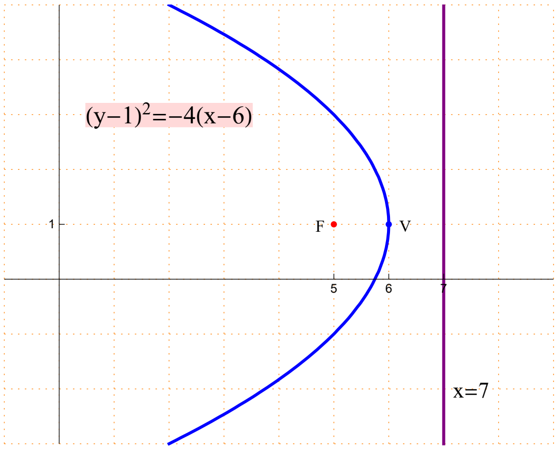
The equation of the parabola has the form $(y-k)^2 = -4a(x-h)$

The vertex is $V(6,1)$, hence $h = 6$, $k = 1$.

"a" is the distance between V and F, hence $a = 1$.

The standard equation of the parabola is $(y-1)^2 = -4(x-6)$

The equation of the directrix is $x = 7$.



Q.5 Find all the elements of the conic section $y^2 - 4x^2 + 10y + 8x + 17 = 0$ and sketch it.

Solution :

$y^2 - 4x^2 + 10y + 8x + 17 = 0$

$y^2 + 10y - 4x^2 + 8x = -17$

$y^2 + 10y - 4(x^2 - 2x) = -17$

By completing the square.

$(y^2 + 10y + 25) - 4(x^2 - 2x + 1) = -17 + 25 - 4$

$(y + 5)^2 - 4(x - 1)^2 = 4$

$$\frac{(y+5)^2}{4} - \frac{4(x-1)^2}{4} = 1$$

$$\frac{(y+5)^2}{4} - \frac{(x-1)^2}{1} = 1$$

The conic section is Hyperbola.

The center is $P = (1, -5)$

$a^2 = 1 \implies a = 1$

$b^2 = 4 \implies b = 2$

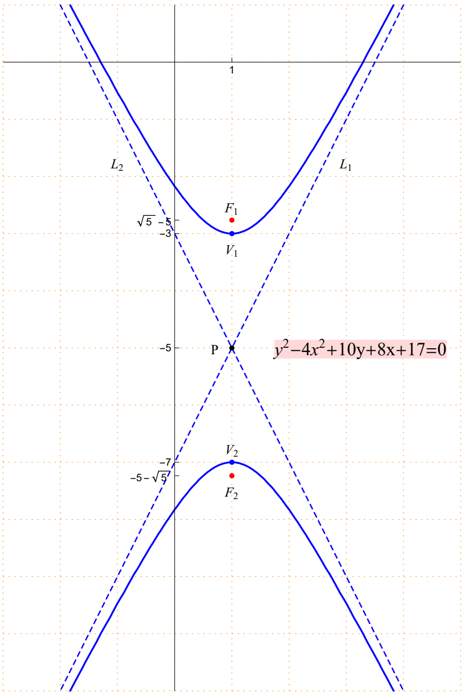
$c^2 = a^2 + b^2 = 1 + 4 = 5 \implies c = \sqrt{5}$

The vertices are $V_1 = (1, -3)$ and $V_2 = (1, -7)$

The foci are $F_1 = \left(1, -5 + \sqrt{5}\right)$ and $F_2 = \left(1, -5 - \sqrt{5}\right)$

The equations of the asymptotes are $L_1 : \ (y + 5) = 2(x - 1)$

and $L_2 : \ (y + 5) = -2(x - 1)$



1
L_2
L_1
F_1
$\sqrt{5}-5$
-3
V_1
-5
P
$y^2-4x^2+10y+8x+17=0$
V_2
-7
$-5-\sqrt{5}$
F_2

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Solution of the Second Mid-Term Exam
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Q.1 Compute the integrals :

(a) $\displaystyle\int 2x(x^2+1)^7 \; dx$

(b) $\displaystyle\int x^2 \cos\left(x^3\right) \; dx$

(c) $\displaystyle\int x^2 \ln x \; dx$

(d) $\displaystyle\int (x+1)e^x \; dx$

(e) $\displaystyle\int \frac{1}{x^2+6x+10} \; dx$

(f) $\displaystyle\int \frac{x+2}{(x-2)(x-4)} \; dx$

Solution :

(a) $\displaystyle\int 2x(x^2+1)^7 dx = \int (x^2+1)^7 (2x) dx = \frac{(x^2+1)^8}{8} + c$

Using the formula $\displaystyle\int [f(x)]^n f'(x) \; dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$

(b) $\displaystyle\int x^2 \cos\left(x^3\right) \; dx = \frac{1}{3}\int \cos\left(x^3\right) \; (3x^2) \; dx = \frac{1}{3}\sin\left(x^3\right) + c$

Using the formula $\displaystyle\int \cos\left(f(x)\right) f'(x) \; dx = \sin\left(f(x)\right) + c$

(c) $\displaystyle\int x^2 \ln x \; dx$

Using integration by parts

$u = \ln x \qquad dv = x^2 \; dx$

$du = \dfrac{1}{x} \; dx \qquad v = \dfrac{x^3}{3}$

$$\int x^2 \ln x \; dx = \frac{x^3}{3}\ln x - \int \frac{1}{x}\frac{x^3}{3} \; dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 \; dx = \frac{x^3}{3}\ln x - \frac{1}{3}\frac{x^3}{3} + c$$

(d) $\int (x+1)e^x \; dx$

Using integration by parts :

$u = x+1 \qquad dv = e^x \; dx$
$du = 1 \; dx \qquad v = e^x$

$$\int (x+1)e^x \; dx = (x+1)e^x - \int e^x \; dx = (x+1)e^x - e^x + c = xe^x + c$$

(e) $\int \frac{1}{x^2+6x+10} \; dx = \int \frac{1}{(x^2+6x+9)+1} \; dx = \int \frac{1}{(x+3)^2+1} \; dx$

$= \tan^{-1}(x+3) + c$

Using the formula $\int \frac{f'(x)}{a^2+[f(x)]^2} \; dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + c$, where $a > 0$

(f) $\int \frac{x+2}{(x-2)(x-4)} \; dx$

Using the method of partial fractions

$\frac{x+2}{(x-2)(x-4)} = \frac{A_1}{x-2} + \frac{A_2}{x-4}$

$x+2 = A_1(x-4) + A_2(x-2)$

Put $x = 2$:

$2+2 = A_1(2-4) \implies 4 = -2A_1 \implies A_1 = -2$

Put $x = 4$:

$4+2 = A_2(4-2) \implies 6 = 2A_2 \implies A_2 = 3$

$\int \frac{x+2}{(x-2)(x-4)} \; dx = \int \left(\frac{-2}{x-2} + \frac{3}{x-4}\right) \; dx$

$= -2\int \frac{1}{x-2} \; dx + 3\int \frac{1}{x-4} \; dx = -2\ln|x-2| + 3\ln|x-4| + c$

Q.2 (a) Sketch the region R_1 determined by the curves

$$y = x^2 - 1 \; , \; y = -1 \; , \; x = 1$$

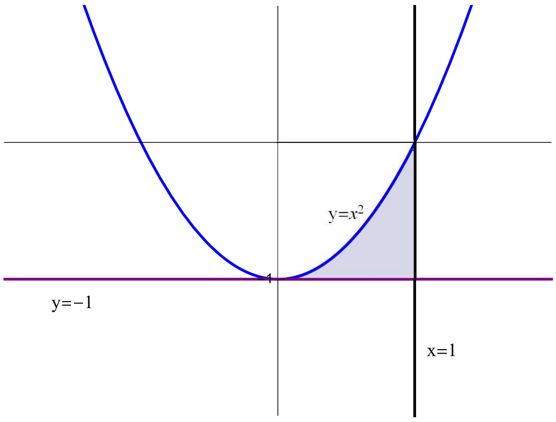
(b) Find the area of the region R_1 described in part (a) .

Solution :

(a) $y = x^2 - 1$ is a parabola opens upwards with vertex $(0, -1)$

$y = -1$ is a straight line parallel to the x-axis and passes through $(0, -1)$

$x = 1$ is a straight line parallel to the y-axis and passes through $(1, 0)$



(b) Area $= \int_0^1 \left[(x^2-1)-(-1)\right]\ dx = \int_0^1 x^2\ dx$

$= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

Q.3 (a) Sketch the region R_2 determined by the curves

$$y = x^2\ ,\ x = 2\ ,\ y = 0$$

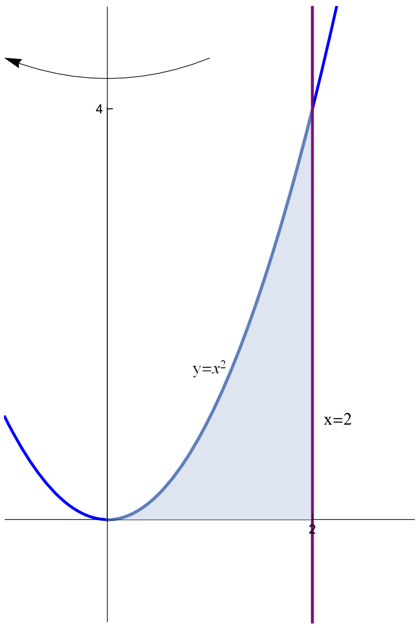
(b) Find the volume of the solid generated by rotating the region R_2 in part (a) about the y-axis .

Solution :

(a) $y = 0$ is the x-axis

$y = x^2$ is a parabola opens upwards with vertex $(0, 0)$

$x = 2$ is a straight line parallel to the y-axis and passes through $(2, 0)$



(b) Using Cylindrical Shells method :

Volume $= 2\pi \int_0^2 x\left(x^2\right)\ dx = 2\pi \int_0^2 x^3\ dx$

$= 2\pi \left[\frac{x^4}{4}\right]_0^2 = 2\pi \left[\frac{2^4}{4} - \frac{0^4}{4}\right] = 2\pi \left[\frac{16}{4}\right] = 8\pi$

Another solution : Using Washer Method

$y = x^2 \implies x = \sqrt{y}$

Volume $= \pi \int_0^4 \left[(2)^2 - (\sqrt{y})^2\right]\ dy = \pi \int_0^4 (4-y)\ dy$

$= \pi \left[4y - \frac{y^2}{2}\right]_0^4 = \pi \left[\left(4 \times 4 - \frac{4^2}{2}\right) - (0-0)\right] = \pi\,(16-8) = 8\pi$