SOLUTION TO M-107 MID TERM EXAM., NOV. 8, 2020

Solution to **Q1** [Marks: 4]

Consider the augmented matrix and apply Gauss-Jordan elimination method to obtain:

$$\begin{pmatrix} 2 & 2 & -2 & 4 \\ 3 & 5 & 1 & -8 \\ -4 & -7 & -2 & 13 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 3 & 5 & 1 & -8 \\ -4 & -7 & -2 & 13 \end{pmatrix} \xrightarrow{(-3)R_1+R_2;\ 4R_1+R_3}$$

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 4 & -14 \\ 0 & -3 & -6 & 21 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -7 \\ 0 & -3 & -6 & 21 \end{pmatrix} \xrightarrow{3R_2+R_3}$$

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-R_2+R_1} \begin{pmatrix} 1 & 0 & -3 & 9 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let

$$x_3 = t. \ \Rightarrow x_2 + 2x_3 = -7 \Rightarrow x_2 = -7 - 2x_3 = -7 - 2t$$

$$x_1 - 3x_3 = 9 \Rightarrow x_1 = 9 + 3x_3 = 9 + 3t.$$

Thus, we have:

$$x_1 = 9 + 3t, \quad x_2 = -7 - 2t, \quad x_3 = t$$

Solution to Q2 [Marks: 5+3=8]

(a)

$$(A|I_3) = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1/6 & 1/3 & -7/6 \\ 0 & 1 & 0 & 1/6 & 1/3 & -1/6 \\ 0 & 0 & 1 & 1/3 & -1/3 & 2/3 \end{pmatrix} = (I_3|A^{-1})$$

where

$$A^{-1} = \begin{pmatrix} 1/6 & 1/3 & -7/6 \\ 1/6 & 1/3 & -1/6 \\ 1/3 & -1/3 & 2/3 \end{pmatrix}$$

(b) Since $AX = B$ with

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \; and \;\; B = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B = \begin{pmatrix} 1/6 & 1/3 & -7/6 \\ 1/6 & 1/3 & -1/6 \\ 1/3 & -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 1/6 \begin{pmatrix} 1 & 2 & -7 \\ 1 & 2 & -1 \\ 2 & -2 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 1/6 \begin{pmatrix} -12 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Hence

$$x = -2, \; y = 0, \; z = 1$$

Solution to Q3 [Marks: 4+4=8]

(a) Let

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 & -2 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} -\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)[-\lambda(2-\lambda)+1]$$

$$= (3-\lambda)(\lambda^2 - 2\lambda + 1) = (3-\lambda)(\lambda-1)^2$$

Since $A - \lambda I$ is not invertible if and only if $|A - \lambda I| = 0$ if and only if $\lambda = 3$ and $\lambda = 1$.

(b)

$$A = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{pmatrix} \; and \; |A| = 10.$$

Therefore, applying Cramer's Rule we get:

$$x = 1/10 \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = 8/10 = 4/5$$

$$y = 1/10 \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -15/10 = -3/2$$

$$z = 1/10 \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -16/10 = -8/5$$

Solution to Q4 [Marks: 2+2=4]

(a) Sum of vectors $\langle 1, -1, 2\rangle + \langle 3, 2, 1\rangle = \langle 4, 1, 3\rangle$. According to the given condition we have $\langle 2, 1, m\rangle \cdot \langle 4, 1, 3\rangle = 0 \implies 8 + 1 + 3m = 0$ implying $m = -3$

(b) Given $P(1, 1, 1)$ and $Q(3, 5, 4)$. So, $\overrightarrow{PQ} = \langle 2, 4, 3\rangle$. Therefore, the work done $W = \vec{a} \cdot \overrightarrow{PQ} = \langle 4, 7, 4\rangle \cdot \langle 2, 4, 3\rangle = 8 + 28 + 12 = 48$.

Solution to Q5 [Marks: 3+3=6]

(a) The vector

$$\vec{a} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & -1 \\ 2 & 1 & 4 \end{vmatrix} = \vec{i} + 2\vec{j} - \vec{k}$$

is perpendicular to both $\vec{u}$ and $\vec{v}$.

Hence, we have the direction vector $\vec{a} = \langle 1, 2, -1 \rangle$. Therefore, the parametric equations of the required line l through the point $P(1, 3, 0)$ are:

$$x = 1 + t, \quad y = 3 + 2t, \ z = -t$$

(b) Here $\overrightarrow{PQ} = \langle -1, -2, 2 \rangle$. Given $\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k} = \langle 3, -1, 2 \rangle$.

So, we have

$$\overrightarrow{PQ} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & 2 \\ 3 & -1 & 2 \end{vmatrix} = -2\vec{i} + 8\vec{j} + 7\vec{k}$$

which is normal to the plane. Therefore, with $P(1, 0, -2)$, the equation of the plane is

$$-2(x - 1) + 8(y - 0) + 7(z + 2) = 0 \Rightarrow -2x + 8y + 7z + 16 = 0$$