Ch(2): Markov Chain.

(1) Definition:

Ω = Sample space.

$X : \Omega \longrightarrow \mathbb{R}$ random variable.

I : time interval

S : state space, $S \subseteq \mathbb{R}$.

* $X : \Omega \times I \longrightarrow S$ is called a stochastic process.

$(\omega, t) \longrightarrow X_t(\omega)$

* $I = \{0, 1, 2, 3, \dots\}$.

the stochastic process X is called discrete time stochastic process.

we denote $X = (X_n)_{n=0,1,2\dots}$

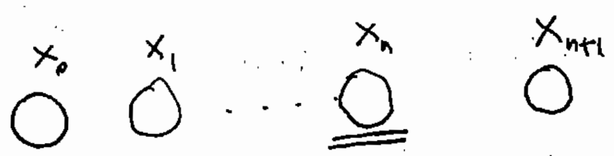
* The state space of X is countable.

$S = \{1, 2, 3, 4, \dots\}$.

* A discrete time stochastic process X is called a Markov chain if:

$$\mathbb{P}\left(X_{n+1} = i \mid X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\right) = \mathbb{P}\left(X_{n+1} = i \mid X_n = i_n\right)$$

for $i, i_1, \dots, i_n \in S$.



$$P_{ij} = \mathbb{P}\left(X_{n+1} = j \mid X_n = i\right)$$

$\mathbb{P} = (P_{ij})_{i,j=0,1,\dots}$ is called a one step transition matrix.

Example: $X_n = \begin{cases} 0 & \text{if it rains in day } n. \\ 1 & \text{if not.} \end{cases}$

$$\mathbb{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$$

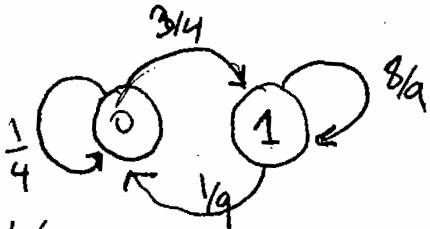
$$p_{00} = \mathbb{P}(X_{n+1} = 0 \mid X_n = 0) = \frac{1}{4}.$$

$$p_{10} = \mathbb{P}(X_{n+1} = 0 \mid X_n = 1) = \frac{1}{9}.$$

$$p_{01} = \mathbb{P}(X_{n+1} = 1 \mid X_n = 0) = 3/4$$

$$p_{11} = \mathbb{P}(X_{n+1} = 1 \mid X_n = 1) = 8/9$$

$$\mathbb{P} = \begin{pmatrix} 1/4 & 3/4 \\ \frac{1}{9} & 8/9 \end{pmatrix}.$$



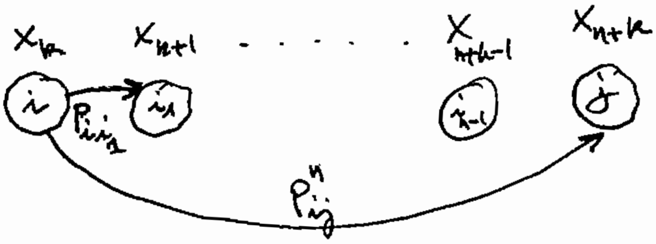
② **Properties:**

(2.1) The one step transition matrix $\mathbb{P}$ is a stochastic matrix:

$$\forall i: \quad \sum_{j \in S} p_{ij} = 1.$$

(2.2) n-step transition matrix:

$$\mathbb{P}^n_{ij} = \mathbb{P}(X_{n+k} = j \mid X_k = i).$$



Example:

$$X^n = \begin{cases} 0 & \text{if it rains on day } n. \\ 1 & \text{if not} \end{cases}$$

$$P_{01}^2 = P(X_{k+2} = 1 \mid X_k = 0).$$

$P^{(n)} = \left(p_{ij}^n\right)$ is called the n-step transition matrix.

Chapman-Kolmogorov equation:

$$\boxed{P^{(n+m)} = P^{(n)} \cdot P^{(m)}}$$

Proof:

$$p_{ij}^{n+m} = P(X_{n+m} = j \mid X_0 = i)$$

$$= \sum_{k=0}^{\infty} P(X_{n+m} = j;\ X_n = k \mid X_0 = i).$$

$$= \sum_{k=0}^{\infty} \underbrace{P(X_{n+m} = j \mid X_n = k;\ X_0 = i)} \cdot P(X_n = k \mid X_0 = i).$$

$$= \sum_{k=0}^{\infty} P(X_{n+m} = j \mid X_n = k)\, P(X_n = k \mid X_0 = i)$$

$$= \sum_{k=0}^{\infty} P_{kj}^m\, P_{ik}^n = \left(P^{(n)} \cdot P^{(m)}\right)_{ij}.$$

Example: $X_n = \begin{cases} 1 & \text{— rains} \\ 0 & \text{if not.} \end{cases}$

$$P^{(1)} = P = \begin{pmatrix} 1/4 & 3/4 \\ \frac{1}{6} & 5/6 \end{pmatrix}.$$

$* \quad P(X_{n+2} = 0 \mid X_n = 1) = ? = P^2_{10}$

we need to Compute the 2-step transition matrix $P^{(2)}$.

$$P^{(2)} = P^{(1+1)} = P^{(1)} \cdot P^{(1)} = P \cdot P = P^2.$$

$$P^2 = \begin{pmatrix} 1/4 & 3/4 \\ 1/6 & 5/6 \end{pmatrix} \begin{pmatrix} 1/4 & 3/4 \\ 1/6 & 5/6 \end{pmatrix} = \begin{pmatrix} 3/16 & \boxed{13/16} \\ \boxed{\frac{13}{72}} & \frac{59}{72} \end{pmatrix}$$

$$P^2_{10} = \frac{13}{72}.$$

$$P^2_{01} = \frac{13}{16} = P(X_{n+2} = 1 \mid X_n = 0).$$

Example: let a Markov chain $(X_n)_{n=0,1,\dots}$ with transition matrix: $S = \{0, 1\}$.

$$P = \begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{pmatrix}$$

Compute: $P(X_4 = 1 \mid X_3 = 0) = ?$

$P(X_4 = 0 \mid X_2 = 1) = ?$

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* $P(X_4 = 1 \mid X_3 = 0) = P_{01} = 0.8$.
* $P(X_4 = 0 \mid X_2 = 1) = P^2_{10}$.

$$P^{(2)} = P^2 = \begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{pmatrix}\begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.36 & 0.64 \\ 0.32 & 0.68 \end{pmatrix}$$

$P(X_4 = 0 \mid X_2 = 1) = 0.32$.

2.3 Mass function of X_n:

$$P(X_n = i) = \sum_{j \in S} P(X_n = i, X_k = j)$$

$$= \sum_{j \in S} \underbrace{P(X_n = i \mid X_k = j)} P(X_k = j).$$

$$= \sum_{j \in S} P^{n-k}_{ji} P(X_k = j).$$

we denote $\mu = f_{X_0}$, the mass function of X_0.

$$P(X_n = i) = \sum_j P^n_{ji} \mu(j) = (\mu . P^{(n)})(i).$$

Ex: let $P = \begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{pmatrix}$. Compute $P(X_2 = 1)$, $E(X_2)$. $P(X_2 = 0)$.

and $\mu = (0.7, 0.3)$.

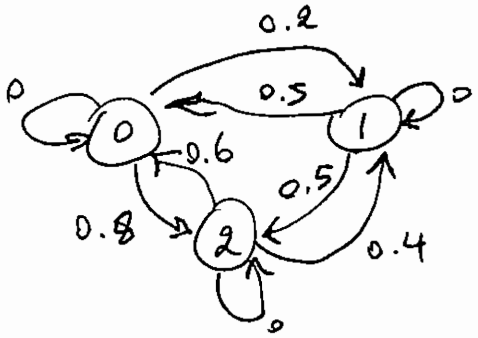
$P(X_2 = 1) = (\mu . P^{(2)})(1)$; $\mu . P^{(2)} = (0.7 \quad 0.3)\begin{pmatrix} 0.36 & 0.64 \\ 0.32 & 0.68 \end{pmatrix}$

$= 0.652$.

$P(X_2 = 0) = 0.348$. $\qquad = (0.348 \quad 0.652)$

$E(X_2) = (1)\, 0.652 + (0).\, 0.348$

Example: let the Markov chain X_n:



let $\mu = (0.1, 0.1, 0.8)$.

Compute: $E(X_1)$, $P(X_2 = i)$, $i = 0,1,2$, $E(X_2)$,
$P(X_1 \geq X_0)$, $P(X_2 < X_1)$.

$$P^{(1)} = P = \begin{pmatrix} 0 & 0.2 & 0.8 \\ 0.5 & 0 & 0.5 \\ 0.6 & 0.4 & 0 \end{pmatrix}.$$

* $$E(X_1) = (0) f_{X_1}(0) + (1) f_{X_1}(1) + (2) f_{X_1}(2)$$

$$\cancel{= (0)}$$

$$f_{X_1} = \mu P = \begin{pmatrix} 0.1 & 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0 & 0.2 & 0.8 \\ 0.5 & 0 & 0.5 \\ 0.6 & 0.4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.53 & 0.34 & 0.13 \end{pmatrix}.$$

$$E(X_1) = (0)\,0.53 + (1)\,0.34 + (2)\,0.13 = 0.6.$$

* $$f_{X_2} = \mu . P^{(2)} = f_{X_1} . P = \begin{pmatrix} 0.53 & 0.34 & 0.13 \end{pmatrix} \begin{pmatrix} 0 & 0.2 & 0.8 \\ 0.5 & 0 & 0.5 \\ 0.6 & 0.4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.248 & 0.158 & 0.594 \end{pmatrix}.$$

$$E(X_2) = (0)\,0.248 + (1)\,0.158 + (2)\,0.594$$
$$= 1.346.$$

* $P(X_1 \geq X_0) = ?$

$$f_{X_1|X_0=i}(j) = \frac{f_{X_0,X_1}(i,j)}{f_{X_0}(i)}$$

$$f_{X_0,X_1}(i,j) = f_{X_1|X_0=i}(j) \cdot f_{X_0}(i)$$

$$= P_{ij}\,\mu(i).$$

$$P(X_1 \geq X_0) = \sum_{i=0}^{2} P(X_1 \geq X_0 \mid X_0 = i)\, P(X_0 = i).$$

X, Y 2 discrete r.vs.

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

$$\boxed{P(A) = \sum_{i \in S} P(A \mid X = i)\, P(X = i)}$$

$$= \sum_{i=0}^{2} P(X_1 \geq i \mid X_0 = i)\, P(X_0 = i)$$

$$= \sum_{i=0}^{2} \sum_{j=i}^{2} \underbrace{P(X_1 = j \mid X_0 = i)}\, P(X_0 = i)$$

$$= \sum_{i=0}^{2} \sum_{j=i}^{2} P_{ij}\,\mu(i)\cdots$$

$$= \sum_{j=0}^{2} P_{0j}\,\mu(0) + \sum_{j=1}^{2} P_{1j}\,\mu(1) + \sum_{j=2}^{2} P_{2j}\,\mu(2).$$

$$= (P_{00} + P_{01} + P_{02})\mu(0) + (P_{11} + P_{12})\mu(1) + P_{22}\,\mu(2).$$

$$= (1)(0.1) + (0.5)(0.1) + (0)\cdot(0.8)$$

$$= 0.15.$$

* $P(X_2 < X_1) = 1 - P(X_2 \geq X_1)$

$$= 1 - \sum_{i=0}^{2} \sum_{j=i}^{2} P_{ij}\, f_{X_1}(i)$$

* $P(X_2 < X_0) = ? = 1 - \sum_{i=0}^{2} \sum_{j=i}^{2} P^{2}_{ij}\,\mu(i).$

Exercise:

Let the Markov chain $(X_n)_{n\geq 0}$ with transition matrix: $P = \begin{pmatrix} 0.1 & 0.9 \\ 0.4 & 0.6 \end{pmatrix}$

and $\mu = f_{X_0} = (0.5 ; 0.5)$.

$0.5 = P(X_0 = 0)$; $0.5 = P(X_0 = 1)$.

Compute f_{X_2} and $E(X_2)$?

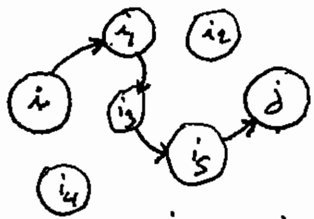
$f_{X_2} = \mu . P^2 = (\mu . P) . P$

$= (0.25 \quad 0.75) . P = (\cancel{0.275}\ 0.325 \quad 0.675)$.

$E(X_2) = (0) 0.325 + (1) 0.675 = 0.675$.

2.4 Classification of states:

* we say that state j is <u>accessible</u> from state i if $P^n_{ij} > 0$ for some $n \geq 0$.



we write $i \longrightarrow j$.

* we say that state i <u>communicate</u> with state j if $i \longrightarrow j$ and $j \longrightarrow i$.

we write $i \longleftrightarrow j$.

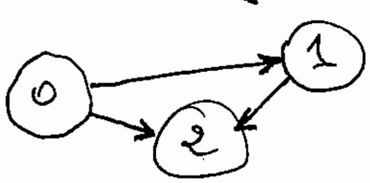
Example: $P = \begin{pmatrix} 0.5 & \boxed{0.5} & \boxed{0} \\ \boxed{0.5} & 0.25 & \boxed{0.25} \\ 0 & \boxed{0.6} & 0.4 \end{pmatrix}$

$P_{10} = 0.5$, $1 \longrightarrow 0$

$P_{21} = 0.6$, $2 \longrightarrow 1$

$P_{20} = 0$.

$2 \longrightarrow 0$



$P_{01} = 0.5 > 0$ $\quad 0 \longrightarrow 1$

$P_{02} = 0$

$P_{12} = 0.25 > 0$ $\quad 1 \longrightarrow 2$

$0 \longrightarrow 2$

$0 \longleftrightarrow 1$. $\quad 1 \longleftrightarrow 2$.

$0 \longleftrightarrow 2$

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$P_{01} > 0$, $P_{12} > 0 \Rightarrow \underline{P_{02}^{2} > 0}$.

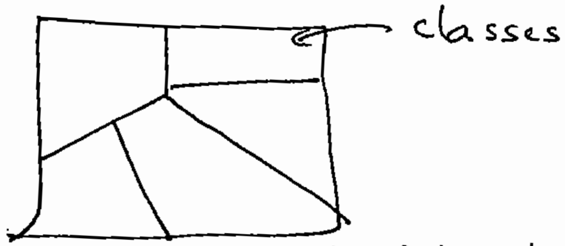
* Equivalence relation:

<u>A set</u> : $x, y \in A$,

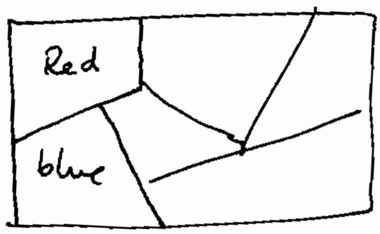
$x \sim y$ is an equivalence relation if:

* $x \sim x$. (reflexive property).
* $x \sim y$, $y \sim z$, then $x \sim z$ (transition property).

* $\longleftrightarrow$ is an equivalence relation.



Ex: let A the set of Colored items.
$x \sim y$ if x and y have the same Color.



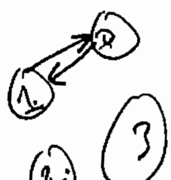
let the states $0, 1, 2, \dots, n$.

$\{0,1,2\}$; $\{3,4\}$, $\{5,7,9\}$, $\{\cdots\}$.

$1 \not\longleftrightarrow 4$, $4 \not\longleftrightarrow 5$.

* Example: let
$$P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Find the classes of this Markov chain?



$P_{01} > 0$; $0 \longrightarrow 1$ $\qquad 0 \longleftrightarrow 1$.

$P_{10} > 0$, $1 \longrightarrow 0$

$P_{02} = 0$ $\qquad 0 \not\longleftrightarrow 2$ $\qquad 1 \not\longleftrightarrow 2$.

$P_{12} = 0$. $\qquad P_{23} = 0.25$ $\qquad 2 \longrightarrow 3$.

$P_{03} = 0$ $\qquad 3 \not\longrightarrow 2$. $\therefore$

$P_{13} = 0$ $\qquad P_{32} = 0$. Classes are: $\{0,1\}, \{2\}, \{3\}$.

* state i is absorbing if $P_{ii} = 1$.

* state i is called recurrent if:

  $f_i := P\left(\text{Markov chain will ever enter } i \mid X_0 = i\right) = 1$

* state i is called transient if:

  $f_i < 1$.

Theorem:

state i is recurrent if $\sum_n P_{ii}^n = \infty$,

state i is transient if $\sum_n P_{ii}^n < \infty$.

Example: Let $P = \begin{pmatrix} 0.2 & 0.8 \\ 0 & 1 \end{pmatrix}$.

Determine the recurrent and transient states.

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state ① is absorbing, ① is recurrent.

$$0 \longrightarrow 1$$

$$\sum_n P_{00}^n = ? \qquad P^n = ?$$

$$\varphi(\lambda) = (\lambda-1)^1(\lambda+1)^1(\lambda+2)^1$$

$1, -1, -2.$

$$P = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_k \end{pmatrix}, \qquad P^n = \begin{pmatrix} a_1^n & & & 0 \\ & a_2^n & & \\ & & \ddots & \\ 0 & & & a_k^n \end{pmatrix}$$

$$\varphi(\lambda) = (\lambda-1)^2(\lambda+1)^1$$

$1, -1.$

$P =$

eigenvalues are simple (order 1).

$$P = M D M^{-1}, \quad P^n = M D^n M^{-1}.$$

$$P^n = \begin{pmatrix} (0.2)^n & 1-(0.2)^n \\ 0 & 1^n \end{pmatrix}.$$

$$\sum_{n\geq 0} P_{00}^n = \sum_{n \geq 0} (0.2)^n = \frac{1}{1-0.2} = \frac{1}{0.8} < \infty.$$

$\therefore$

state 0 is transient.

$$P^2 = \begin{pmatrix} 0.2 & 0.8 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (0.2)^2 & 1-(0.2)^2 \\ 0 & 1 \end{pmatrix}.$$

$$P^3 = \dots\dots, \quad P^n = ($$

2.5 Limiting Probabilities:

$(X_n)_{n=0,1,\dots}$

Under which conditions, $\lim_{n\to\infty} P_{ij}^n$ exists.

Theorem: * For an irreducible (one class) ergodic (recurrent) Markov chain (X_n); $\lim_{n\to\infty} P_{ij}^n$ exists and independent of i.

$$\pi_j = \lim_{n\to\infty} P_{ij}^n.$$

* $\pi = (\pi_0, \dots, \pi_n)$ is the solution of:

$$\pi = \pi . P, \quad \sum_j \pi_j = 1.$$

* π_j is the long-run proportion of the Markov chain at state i.

Example: let $P = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$.

$\pi = ? = (\pi_0, \pi_1, \pi_2)$.

$$\pi = \pi . P \iff \begin{cases} 0.5\pi_0 + 0.3\pi_1 + 0.2\pi_2 = \pi_0 \\ 0.4\pi_0 + 0.4\pi_1 + 0.3\pi_2 = \pi_1 \\ 0.1\pi_0 + 0.3\pi_1 + 0.5\pi_2 = \pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}.$$

$$\iff \begin{cases} -0.5\pi_0 + 0.3\pi_1 + 0.2\pi_2 = 0 \\ 0.4\pi_0 - 0.6\pi_1 + 0.3\pi_2 = 0 \\ 0.1\pi_0 + 0.3\pi_1 - 0.5\pi_2 = 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}.$$

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$$\pi_0 = \frac{21}{62}, \quad \pi_1 = \frac{23}{62}, \quad \pi_2 = \frac{18}{62}.$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$33.9\% \qquad 37.1\% \qquad 29\%$$

0

1

2

Midt: Joint random variables

$\longrightarrow$ limiting Prob.