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Marginal Analysis, Multi-Plant Firms, and Business Practice: An Example

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Source: *The Quarterly Journal of Economics*, Vol. 69, No. 2 (May, 1955), pp. 253-268

Published by: [Oxford University Press](#)

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# MARGINAL ANALYSIS, MULTI-PLANT FIRMS, AND BUSINESS PRACTICE: AN EXAMPLE\*

*By* FRED M. WESTFIELD

I. Introduction, 253. — II. Allocation of output among plants, neglecting transmission losses, 254. — III. Practical solution of this problem, 258. — IV. Allocation of output, making allowances for transmission losses, 261. — V. Practical solution, making allowances for transmission losses, 265. — VI. Conclusion, 268.

## I

Theoretical economists are often criticized for failing to demonstrate the practical value of their research. Economic literature is almost completely barren of expositions relating how even some of the most basic principles of economic logic find actual use in specific business situations. There exists in fact a widespread belief that the techniques and results of traditional economic theory have little or nothing to contribute to the practical solution of the businessman's concrete problems. It seems appropriate, therefore, to report an interesting example of how the general principles of marginal analysis, so fundamental in theoretical discussions of resource allocation, are actually profitably applied by men of affairs in the electric power generating industry faced with a complex allocation problem.

Typical of the modern electric power generating industry are large integrated companies that operate several geographically separated generating plants and service customers over a wide region frequently encompassing parts of several states. The plants of such a system are interconnected by a complex web of transmission lines so that electricity can be "transported" from any plant to every part of the network.

Over a twenty-four hour period the demand for electricity fluctuates sharply. Periods during which demand is less than 40 per cent of peak demand are not uncommon for even the largest systems. These demand patterns vary somewhat from day to day. In the short run the number and types of available plants are fixed, the price structure is given, and the electricity demanded must be delivered to consumers regardless of whether out-of-pocket expenses are covered by revenues. From the economist's as well as from the

\* I am indebted to Professor Paul A. Samuelson for first stimulating my interest in this inquiry and for subsequent help and encouragement.

operator's point of view, the important short-run decision is how to allocate the demanded output among the alternative plant capacities so as to minimize costs. The stakes are high. Even a small percentage saving in costs averaging, say, \$20 per hour amounts to an annual saving of about \$175,000.

Until recently losses of electricity during transmission (transport costs) were generally ignored in the solution of the problem, except in isolated cases where they were obviously great. During World War II, however, practical methods of determining the losses for routine operating purposes were developed by electrical engineers; and in 1951 the American Gas and Electric Company, a system operating over forty interconnected plants and serving customers in some 2300 communities of seven states, was pioneering in using these loss data to determine least cost production schedules for their plants. Taking explicit cognizance of these losses, equivalent to abandoning the economist's familiar "no-transport-costs" assumption, leads to estimated additional cost savings of close to \$150,000 per year in the operation of this system.

Paradoxically, engineers, not economists, were called upon to analyze and solve this resource allocation problem. They worked out the correct solutions long known and understood by economists. Showing no familiarity with economic literature, they "discovered" marginal analysis after some stumbling and then proceeded to apply its principles. Their theoretical expositions lack the rigor and sophistication to which economists are accustomed; but their techniques and devices for making the discovered optimal solutions operational within the firms of the industry exhibit adroitness and expediency to which economists are unaccustomed.

We shall report, in turn, the solutions of the problem, first neglecting transmission losses, then introducing them explicitly. For each case there will be developed: (i) the economist's theoretical solution which corresponds, except for detail, to the engineer's conclusions, and (ii) the industry's methods of utilizing the theoretical results.

## II

How a multi-plant firm must allocate output among plants, in the absence of transport losses and transport costs, so as to minimize costs was carefully developed some years ago in an article by Professor Patinkin and a comment thereon by Professor Leontief.<sup>1</sup> They con-

1. D. Patinkin, "Multi-Plant Firms, Cartels, and Competition," this *Journal*, LXI (Feb. 1947), 173-205. W. Leontief, "Multiple-Plant Firms: Comment,"

sidered a firm with  $n$  plants, each with a total cost curve of the general form

$$C_i = c_i(x_i) + F_i,$$

where  $C_i$  are the total costs of plant  $i$ ,  $c_i(x_i)$  its variable costs as a function of its output  $x_i$ , and  $F_i$  its fixed costs. The total costs of the firm

$$(1) \quad \sum_{i=1}^n C_i + G = C,$$

where  $G$  are fixed expenses not attributable to the operations of any one plant, must be a minimum for each level of total output demanded. This is one of those constrained minimum problems so familiar to economists.<sup>2</sup> The constraint is simply

$$(2) \quad \sum_{i=1}^n x_i = D,$$

$D$  being the total output demanded.

Making the customary continuity assumptions and employing the techniques of the calculus, one can deduce the desired theoretical requirements for minimum costs. The first order conditions for a minimum tell us that if more than one of the firm's plants are in operation, each must be made to produce at a rate such that its marginal costs are equal to the marginal costs of every other plant in operation. And a plant is to stand idle only if its marginal costs when idle are higher than those of the plants in operation. These conditions must be satisfied for every level of the firm's total output.

It is useful for what follows to summarize these results by the expression

$$(3) \quad C'_i = \lambda \quad (i = 1, 2, \dots, n),$$

where  $C'_i$  stands for the marginal costs of the  $i$ th plant, and  $\lambda$  is a Lagrangean multiplier whose value is determined by the amount of

this *Journal*, LXI (Aug. 1947), 650-51. How a monopolist should allocate his output among several separated markets in the absence of transport costs — a similar problem — was carefully worked out by T. O. Yntema, "The Influence of Dumping on Monopoly Price," *Journal of Political Economy* (Dec. 1928), XXXVI, 686-98. Almost every modern economics text might be cited for the solution of problems that have much in common with that under discussion here.

2. Cf., for example, P. A. Samuelson, *Foundations of Economic Analysis*, pp. 57 ff.

total output demanded. It is to be understood that the equality signs must be replaced with a "greater than" sign for idle plants.

In general, these conditions are not enough. The second order conditions require that (i) not more than one plant with "forward-falling" marginal costs should ever be in operation. Further, (ii) if one of the several plants in operation is producing on a falling part of its marginal cost curve, then, at these outputs, the rate of decrease of its marginal cost curve must be smaller in absolute value than the rate of increase of the horizontally summed marginal cost curves — a combined marginal cost curve — of all the other plants not standing idle.<sup>3</sup>

The mathematically minded reader will note that these results are readily obtained by discovering the conditions under which the Lagrangean expression

$$C - \lambda(\Sigma x_i - D)$$

subject to the constraint, takes on a relative minimum. By setting the first derivatives with respect to each  $x_i$  equal to zero one obtains the set of equations (3). Boundary minima will introduce the inequalities.

The second order conditions require that the expression

$$(4) \quad \prod_{i=1}^k C_i'' \sum_{i=1}^k \frac{1}{C_i''} > 0 \quad (k = 2, 3, \dots, m \leq n),$$

where  $C_i''$  refers to the slope of the marginal cost curve for plant  $i$  evaluated at the optimum output, be satisfied for all  $k$ , regardless how we number the  $m$  ( $\leq n$ ) plants in operation.<sup>4</sup> Part (i) of the second order conditions stated in words above is deduced from the

3. The next two paragraphs may be omitted without loss of continuity by those not interested in the mathematical derivation of the results.

4. The principal minors of the determinant

$$\Delta = \begin{vmatrix} C_i'' \delta_{ij} & 1_j \\ 1_j & 0 \end{vmatrix}, \quad \begin{matrix} \delta_{ij} = 0 \text{ for } i \neq j, \\ \delta_{ij} = 1 \text{ for } i = j, \end{matrix}$$

1<sub>i</sub>, 1<sub>j</sub> = column and row, respectively, of ones,

as well as the determinant itself must be negative (*cf.* Samuelson, *op. cit.*, pp.

362 ff.). Evaluating  $\Delta$  by the Laplace Development:  $\Delta = - \sum_{i=1}^m A_{ii}$ , where each  $A_{ii}$  is a nonvanishing cofactor of an element (i.e., of a diagonal element) in the determinant  $|C_{ij}'' \delta_{ij}|$ , and noting that

$$\bar{A}_{ii} = \frac{C_1'' C_2'' \dots C_i'' \dots C_m''}{C_i''}$$

fact that this expression must hold for  $k = 2$ ; part (ii), from the fact that it must hold for  $k = m$ .<sup>5</sup>

There may be several allocations satisfying our conditions. To determine which one of these is "best," it is necessary to compare total variable costs at the various allocations and select the one with the lowest. The optimum allocations are, of course, independent of all fixed costs.

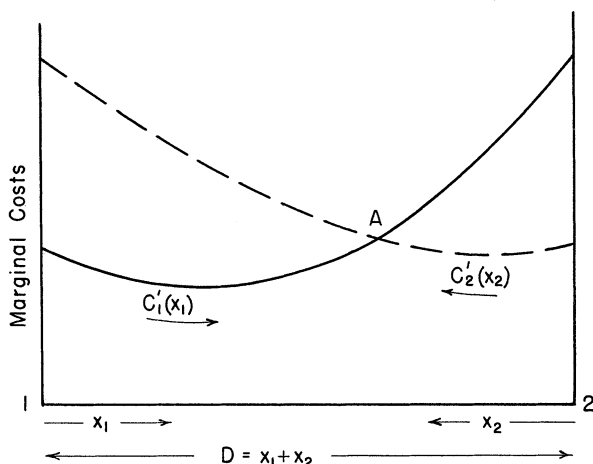


FIGURE I

The output of plant 1 ( $x_1$ ) is read off from left to right, that of plant 2 ( $x_2$ ) from right to left. Total output to be delivered is represented by  $D$ , equal in the absence of transport losses to the sum of  $x_1$  and  $x_2$ . The output allocation among plants corresponding to point A, where the marginal costs of the two plants are equal ( $C_1' = C_2' = \lambda$ ), is the least cost allocation. The combined variable costs for the two plants, represented by the sum of the areas under their respective marginal cost curves, is seen to be greater for every other allocation.

it is readily seen that the expression (4) is a statement of the conventional secondary conditions for a constrained minimum.

5. If, say,  $C_1'' < 0$ , then in (4)  $\prod_{i=1}^m C_i'' < 0$ ; but  $\sum_{i=1}^{k=m} \frac{1}{C_i''} < 0$  must then be satisfied for  $k=m$ , which assures that it will also be satisfied for all other (lower)  $k$ . This condition on the summation may be written  $\sum_{i=2}^{k=m} \frac{1}{C_i''} < -\frac{1}{C_1''}$ , which implies  $\sum_{i=2}^m \frac{1}{C_i''} < \left| \frac{1}{C_1''} \right|$ . Hence,  $\frac{1}{\sum_{i=2}^m \frac{1}{C_i''}} > |C_1''|$ . But the left-hand side of this

inequality is the slope of the horizontal sum of the rising marginal cost curves of the plants in operation; the right-hand side, the absolute value of the slope of the falling marginal cost curve.

If a firm has only two plants it is simple to show the theoretical conclusions on a familiar diagram (Figure I). Experimentation with different shaped marginal cost curves will convince the reader of the validity of all the conditions for minimum costs that we have stated.

### III

So much for the theoretical economics. How is the problem solved in practice?

In the production of electric power, the output at each plant varies, in the short run, primarily with the energy (heat) input.<sup>6</sup> These relationships between the amount of heat energy provided by the fuel and the corresponding maximum amount of electric energy produced by the equipment in a plant, the production functions, are obtained from engineering data. With the technological characteristics and the price of fuel for each plant known, (short-run) marginal cost functions are readily calculated.<sup>7</sup> They may be written as

$$C'_1 \equiv w_i \frac{1}{dx_i/dv_i} ,$$

where  $w_i$  is the price of fuel in plant  $i$  and  $dx_i/dv_i$  its marginal productivity. The units of  $v_i$  are stated in millions of B.T.U. per hour; of  $w_i$  in dollars per million B.T.U.; and of  $x_i$  in megawatts. The B.T.U. content and therefore the effective price of fuel are obtained from routine analyses.

In order to facilitate the actual job of allocating the demanded electricity or "load" among plants in conformance with our theoretical principles, a Station-Loading Sliderule was developed.<sup>8</sup> This mechanical device permits a centrally located dispatcher, in constant communication with the plants, to calculate quickly the optimum allocations for each demand situation.

The essentials of the sliderule, shown schematically in Figure II, are two unmovable scales, the "marginal fuel rate scale" and the "relative fuel price scale," and as many movable "plant output slides" as there are plants in the system — one for each plant. Marginal fuel rates are the additional amounts of fuel required (in millions of B.T.U.) in a plant to produce extra units of output (in megawatt-

6. In this paper we shall ignore the complications that may arise if some of the system's electricity is generated in hydro-electric plants.

7. M. J. Steinberg and T. H. Smith, *Economy Loading of Power Plants and Electric Systems* (New York, 1943), pp. 141 ff.

8. *Ibid.*, pp. 189 ff.

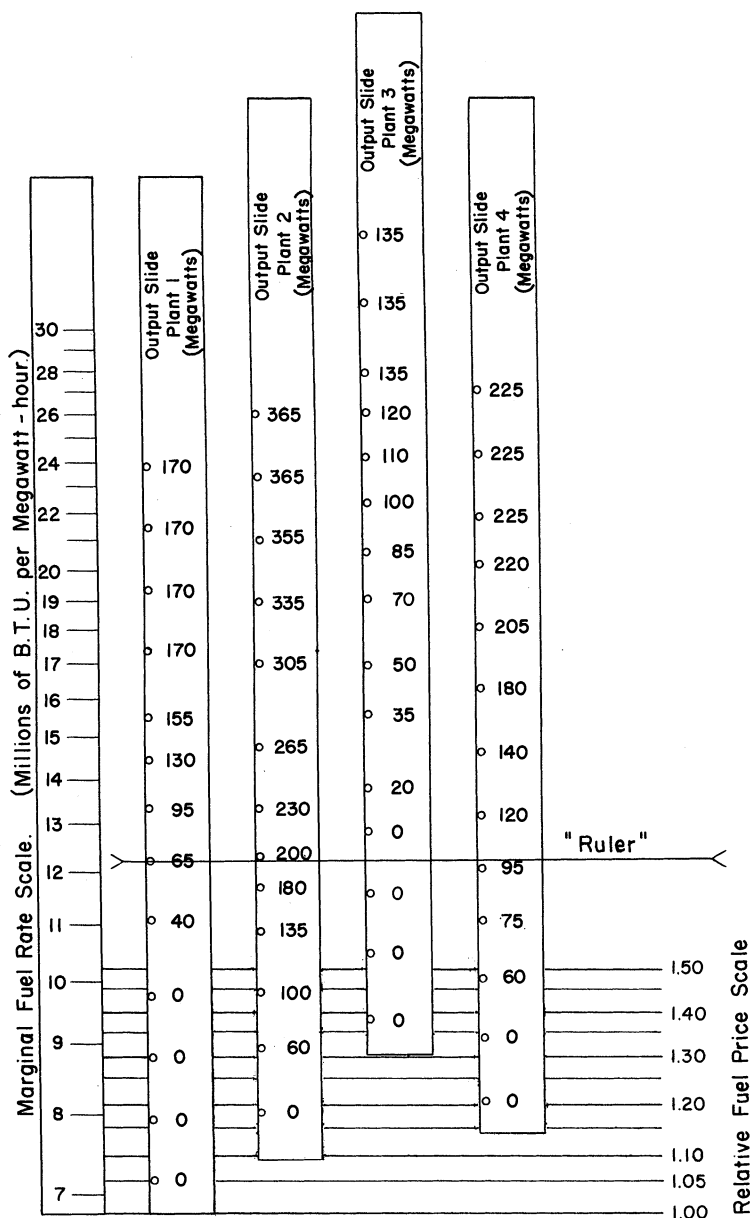


FIGURE II

The Station-Loading Sliderule. Marginal Fuel Rates and Relative Fuel Prices are on logarithmic scales. Plant 1 is the *numéraire* plant. Output is demanded at the rate of 360 megawatts. The optimum allocations to the plants are read off along the ruler.



hours). They are reciprocals of marginal productivities. Relative fuel prices are the prices of fuel at each plant relative to the price of fuel at the plant with the lowest fuel price. We may call the latter plant(s) the *numéraire* plant(s).

The marginal fuel rate scale is calibrated with the logarithms of marginal fuel rates ( $\log dv/dx$ ), starting at the bottom with values somewhat below those actually to be encountered and increasing continuously to some point at the top slightly greater than the largest value to be encountered in the system. Relative fuel prices are also on a logarithmic scale. This scale begins with  $\log 1$  — the index — and runs alongside the marginal fuel rate scale. Each output slide is calibrated in such a way that when the bottom of the slide is lined up with the index on the fuel price scale, its markings show the maximum output rates in the plant that the slide represents corresponding to every marginal fuel rate.

In using the sliderule the dispatcher displaces each of the plant output slides (except that of the *numéraire* plant) from the index to the appropriate point on the relative fuel price scale. Thus, if the price of fuel at the *numéraire* plant is 25¢ per million B.T.U. and at plant 2 the price is 30¢, the bottom of the output slide of plant 2 is moved to 1.2 on the relative fuel price scale. To divide the output among plants so as to "equalize" marginal cost, he merely has to move a "ruler," crossing the output slides horizontally, to that position where the sums of the outputs marked on the slides along the ruler add up to the total demand — whatever it may be at the moment. These outputs of the individual plants satisfy the first order conditions for a minimum which we discussed above.<sup>9</sup>

Instead of equating marginal costs of the plants in operation, the dispatcher using this device actually equates

$$(5a) \quad \log \frac{1}{dx_1/dv_1} + \log \frac{w_1}{w_1} = \log \frac{1}{dx_2/dv_2} + \log \frac{w_2}{w_1} = \dots$$

But this is identically

$$(5b) \quad \log \frac{C_1'}{w_1} = \log \frac{C_2'}{w_1} = \dots$$

9. This sliderule is reminiscent of the one introduced by Edgeworth (*Papers Relating to Political Economy*, II, 52–58) to illustrate Mangoldt's contribution to the comparative cost doctrine in international trade theory. Cf. J. Viner, *Studies in the Theory of International Trade*, pp. 458–67.

A moment's reflection will convince the reader that these equations have the same solutions as set (3).<sup>1</sup> Also, as inspection of Figure II will show, the required inequalities will hold for all idle plants. Using logarithms of marginal costs is convenient when there is only one variable factor (input) because the marginal cost function is simply shifted without change in shape in response to price changes of the factor.

Fortunately, the complexities introduced by decreasing marginal costs can be ignored. The very nature of plant equipment insures that marginal fuel rates and hence marginal costs are increasing functions of output. Proper use of the sliderule will satisfy the first order necessary conditions and rising marginal cost curves at every plant make these conditions also sufficient for minimum cost allocation of output.

It should be noted that no essential complications arise, either in the theoretical solution or in its application, if marginal cost curves of the plants are discontinuous.<sup>2</sup> In fact, in many systems some plants (the most modern and favorably located fuel-pricewise) are always used up to capacity.

#### IV

The foregoing solution to the allocation problem is optimal only if no variable costs or losses of electricity result from transporting the generated electricity from plants to consumers. Typically there are no variable transport costs as such once the system is built; but electrical energy is lost during the process of transmission. Such losses for a system, it was shown as recently as twelve years ago,<sup>3</sup> can be expressed in terms of a quadratic form

$$y = y(x_1, x_2, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n B_{ij} x_i x_j,$$

where  $y$  represents the loss in megawatts and the  $x$ 's, as previously, are the outputs of the plants. Subsequently, methods were developed

1. Just as the equilibrium conditions in consumer theory are independent of monotonic stretchings of the "utility index," so the equilibrium conditions for a multi-plant firm are independent of such a "cost index."

2. Since, corresponding to each marginal fuel rate, the maximum output is marked on the plant's output slide, a discontinuity means that scheduled output for the plant in the neighborhood of the discontinuity will remain unchanged for several different values of the marginal fuel rate. Also, cf. Samuelson, *op. cit.*, pp. 70 ff.

3. E. E. George, "Intrasystem Transmission Losses," *Electrical Engineering (AIEE Transactions)*, Vol. 62 (March 1943), pp. 153-58.

for deriving the loss formula coefficients,  $B_{ij}$ , by means of network analyzers — miniature representations (electrical analogues) of the entire integrated generating and transmission system.<sup>4</sup>

A number of simplifying assumptions are made in the development of the formula and in the estimation of the coefficients. The most significant of these, from the economist's point of view, is that during the normal course of demand variations, the demands for electricity at every point in the system are assumed to vary proportionately. This means that we may think of all the electricity actually sold to many consumers all along the system as if it were delivered to a single consumer located at a fixed delivery point. Should this assumption be unrealistic for a particular system, additional variables may be introduced into the formula, or a different set of coefficients may have to be used for each pattern of demand variations. But these complications will not concern us in this paper.<sup>5</sup>

The essence of the theoretical solution which makes allowances for transmission losses was worked out quite recently by electrical engineers.<sup>6</sup> Again the problem is to minimize the firm's total costs, represented by equation (1), for each level of demanded output. But the simple "conservation-of-product constraint" (equation (2)) is no longer appropriate. There are losses which depend upon how much electricity each plant is producing. The new constraint becomes

$$(2') \quad \sum_{i=1}^n x_i = D + y(x_1, x_2 \dots x_n).$$

It simply says that the total electricity generated by all the plants together must add up to the total quantity demanded by consumers plus the quantity lost in transmission.

Proceeding as before, we can readily discover the necessary and sufficient conditions for minimum cost allocation.<sup>7</sup> The first order

4. J. B. Ward, J. R. Eaton, H. W. Hale, "Total and Incremental Losses in Power Transmission Networks," *AIEE Transactions*, Vol. 69 (Part I, 1950), pp. 626-31.

5. *Ibid.*, p. 629.

6. E. E. George, H. W. Page, and J. B. Ward, "Coordination of Fuel Cost and Transmission Loss by Use of the Network Analyzer to Determine Plant Loading Schedules," *AIEE Transactions*, Vol. 68 (Part II, 1949), pp. 1152-60. L. K. Kirchmayer and G. W. Stagg, "Evaluation of Methods of Co-ordinating Incremental Fuel Costs and Incremental Transmission Losses," *AIEE Transactions*, Vol. 71 (Part III), 1952, pp. 513-20.

7. The Lagrangean expression to be minimized under constraint is

$$C - \mu(\sum x_i - y - D).$$

conditions can be summarized by

$$(7a) \quad C_i' - \mu(1 - y_i) = 0,$$

where  $y_i$  is the marginal transmission loss from plant  $i$  ( $= \partial y / \partial x_i$ ); it is the percentage of an extra unit of power produced at plant  $i$  that is lost during transmission if the output of all other plants remains the same. By rewriting these relations in the form analogous to (3)

$$(7b) \quad \frac{C_i'}{(1 - y_i)} = \mu,$$

where it is again understood that the equal sign must be replaced with a "greater than" sign if the  $i$ th plant is idle, we see, since  $1 - y_i$  represents the marginal percentage of the power produced in plant  $i$  delivered to the consumer, that the new Lagrangean Multiplier  $\mu$  is the marginal delivered cost of electricity — the marginal c.i.f. cost, so to speak.

It is now simple to state these results in words. If more than one plant is being operated, whatever the demanded output may be, each must be operated at a rate such that the marginal delivered cost of its output (the marginal cost of the output generated by it actually reaching the consumer) is equal to the marginal delivered cost of the output of each and every other plant in operation. A plant should be kept idle only if the marginal delivered cost of its output when idle is greater than the marginal delivered cost of those actually operating.

The necessary and sufficient second order conditions are not easily stated in words.<sup>8</sup> However, if we know that all the marginal production cost curves are rising functions of output, the marginal conditions above are sufficient to assure us of a regular minimum solution. As we have indicated, the marginal production cost curves of electric generating plants are by their very nature this type of function.

Marginal transmission losses can also be expressed as marginal transmission (transport) costs. Rewriting equations (7b) in the equivalent form

8. Letting  $\partial^2 y / \partial x_i \partial x_j = y_{ij}$  and using the notation of note 4, p. 256, the necessary and sufficient conditions for the minimum are that the bordered determinant

$$\begin{vmatrix} C_i''\delta_{ij} + \mu y_{ij} & 1_i - y_i \\ 1_j - y_j & 0 \end{vmatrix}$$

and all its principal minors be less than zero. As the quadratic form  $y$  (the loss formula) is intrinsically positive, it is seen that these conditions do hold if all  $C_i''$  are non-negative.

$$(7c) \quad C'_i + \mu y_i = \mu,$$

the marginal delivered cost,  $\mu$ , is seen to be the sum of the marginal production cost,  $C'_i$ , and the marginal transmission loss,  $\mu y_i$  — the marginal transmission loss “priced” at the marginal delivered cost.

Before proceeding with the explanation of how these results are brought to bear in actual practice in scheduling production of the generating plants, it may be helpful to show our theoretical solution graphically for a two-plant firm. For this purpose a diagram different from that used to illustrate the no-transmission-loss-case is convenient. In Figure III the electricity produced by plant 1 is plotted on the horizontal axis, that produced by plant 2 on the vertical. The family of curves labeled  $C^a, C^b, C^c$ , etc., are what I call Constant Outlay Curves. For a given rate of total expenditure by the firm, say  $C^b$ , the curve labeled  $C^b$  shows the maximum rate of output that

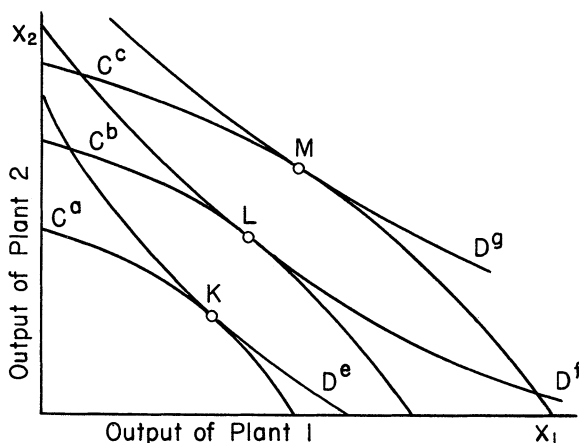


FIGURE III

The points  $K, L, M$ , represent least-cost allocations of demanded output rates  $D^e, D^f, D^g$ , respectively. No allocations can be found for these demand situations which lie on lower Constant Outlay Curves.

can be produced in plant 2 for each specified output rate of plant 1. The further such curves are away from the origin, the greater is the outlay (total expenditure) identified with them. Their slopes are  $-C'_1/C'_2$  — the ratio of marginal production costs in plant 1 to those in plant 2; and if we know that marginal production costs in the plants are increasing functions of output, we can be sure that the Constant Outlay Curves are concave to the origin. The other set of curves,  $D^e, D^f, D^g$ , etc., are what we may call Allocation Possibility Curves. If consumers demand electricity at the rate, say  $D^e$ , the

$D^e$  curve represents all the possible ways of dividing total production of the firm (greater because of losses than the rate of demand) among plants such that in fact the rate  $D^e$  reaches the consumers. The greater the consumer demand, the further away from the origin is the relevant Allocation Possibility Curve. If there are transmission losses, all of these curves must be convex to the origin. Their slopes are  $-(1 - y_1)/(1 - y_2)$  — the ratio at the margin of the percentage of electricity produced in plant 1 reaching the consumer to the percentage of electricity produced in plant 2 reaching the consumer. If transmission losses are neglected,  $y_1 = y_2 = 0$  and the Allocation Possibility Curves are negatively sloped  $45^\circ$  lines.

The optimum allocations are all those points on the graph where the Allocation Possibility Curves touch but do not cross the Constant Outlay Curves. All other allocations for specified demand situations represented by other points on the Allocation Possibility Curves will lie on higher Constant Outlay Curves; hence such allocations cost more. It is seen that the optimum points are precisely the ones specified by the previously stated marginal conditions (equations (7b)).

## V

Attempts to put the theoretical results of this more complex problem into practical operation without modifications would be fraught with difficulties. There are  $n + 1$  nonlinear relations (i.e., equations (2') and (7c)) which would have to be solved simultaneously for the  $n$  plant outputs and for  $\mu$ , for each demand situation. The costs of such computations primarily because of the nonlinearities are likely to be very large.

In actual practice marginal production cost curves of generating plants can be adequately approximated up to maximum output by straight lines

$$C_i' = a_i + b_i x_i, \quad a_i, b_i > 0.$$

Nordin<sup>9</sup> showed this by a careful statistical study and engineers<sup>1</sup> have independently come to the same conclusion. But substitution of these linear relations into equations (7c) will show that this good fortune does not eliminate the difficulties. There are still  $n + 1$  equations, and they are still nonlinear.

Two approximate solutions have been suggested by the engineers.

9. "Note on a Light Plant's Cost Curves," *Econometrica* (July 1947), Vol. 15, pp. 231-35. The findings imply that a rising linear function of output gives significantly better estimates of marginal costs in the plant than Nordin studied than polynomials or a constant.

1. George, Page, and Ward, *op. cit.*, p. 1153. See also the discussion of this paper, pp. 1160-61.

Both have been verified by actual sample calculations to yield allocations extremely close to the theoretical optimum. The first, a linear approximation,<sup>2</sup> is obtained by replacing on the left of each equation (7c) the variable  $\mu$  with a constant  $\gamma$ :

$$(8a) \quad C_i' + \gamma y_i = \mu.$$

Making use of the linear estimates of the marginal cost curves and substituting for  $y_i$  the partial derivatives of the loss formula this becomes

$$(8b) \quad a_i + b_i x_i + \gamma (2 \sum_{j=1}^n B_{ij} x_j) = \mu.$$

The new constant  $\gamma$  is a weighted average value of marginal delivered costs  $\mu$  and is used to "price" the transmission losses. Note that these are linear relations in the unknowns. Even if inequalities replace some equalities for certain values of demanded output, the allocations corresponding to each  $\mu$ , hence by equation (2') for each level of delivered output, may be obtained without difficulties by the use of modern high-speed electronic computing equipment.

The other approximation, now actually used in the American Gas and Electric Company System, is called the "Penalty Factor Method."<sup>3</sup> Rather than "pricing" transmission losses at actual marginal delivered costs ( $\mu$ ), as in the exact solution, or at some average value of marginal delivered costs ( $\gamma$ ), as in the linear approximation, this method charges for them at the marginal production costs of each plant in question ( $C_i'$ ). The equations (7c) then become

$$(9a) \quad C_i' + C_i' y_i = C_i' (1 + y_i) = \mu;$$

or substituting for  $y_i$ ,

$$(9b) \quad C_i' (1 + 2 \sum_{j=1}^n B_{ij} x_j) = \mu.$$

The term  $1 + y_i$  is called the "penalty factor" for plant  $i$ .<sup>4</sup> These penalty factors can be efficiently calculated for each plant under numerous operating conditions of the system.<sup>5</sup>

An advantage of this method is that it permits continued use of the Station-Loading Sliderule, which, as will be recalled, does not require linear approximations or, for that matter, continuity of the

2. *Loc. cit.*

3. L. K. Kirchmayer and G. H. McDaniel, "Transmission Losses and Economic Loading of Power Systems," *General Electric Review*, Vol. 54 (Oct. 1951), pp. 39-46. Also, Kirchmayer and Stagg, *op. cit.*

4. The "penalty factor" consists of the first two terms of the expansion of  $1/1 - y$  in the exact formulation, equations (7b).

5. *The Wall Street Journal*, August 12, 1953, p. 18, reported the purchase of a \$110,000 electronic computer by the American Gas and Electric Company for calculating "penalty factors."

marginal production cost curves. Taking logarithms of the set of equations (9a) and writing it in the form of equations (5a):

$$\begin{aligned} (10) \quad & \log \frac{1}{dx_1/dv_1} + \log \frac{w_1}{w_1} + \log (1 + y_1) \\ & = \log \frac{1}{dx_2/dv_2} + \log \frac{w_2}{w_1} + \log (1 + y_2) = \dots, \end{aligned}$$

one sees that the penalty factors, once known, play a role much like relative fuel prices and can be handled by the dispatcher in a similar manner.

He proceeds by first obtaining a "trial allocation schedule" which neglects transmission losses in precisely the manner described earlier. The output slide for each plant is then moved upward on the relative fuel price scale by the value of its penalty factor corresponding to this trial allocation. A new allocation schedule is read off and each output slide is now displaced by the change in its penalty factor resulting from the change in allocations. This cycle is repeated until for every plant the penalty factors used on the fuel price scale correspond to those penalty factors called for by the allocation schedule obtained with their use. This iterative process converges rapidly because penalty factors are generally small compared to relative fuel prices. Moreover, in determining allocations on an hour-to-hour basis only comparatively small changes in total demand are usually involved. By starting with the allocation schedule of the previous period rather than with the trial schedule obtained by neglecting transmission losses entirely, the changes in penalty factors are even smaller.<sup>6</sup>

While the allocations obtained by either of the two approximate methods do not vary for practical purposes from those obtained by solving the exact nonlinear equations, they may differ markedly from those obtained when transmission losses are neglected. Results of sample calculations, reproduced in Figure IV, for a simplified eight-plant representation of the American Gas and Electric Company's network of almost fifty plants show the estimated additional savings to the firm under the various demand conditions when in determining allocations cognizance is taken of transmission losses. It is estimated that savings in production costs, or, perhaps more appropriately, in

6. It is not clear to the author why the approximate  $(1 + y_i)$  is calculated as the penalty factor rather than the exact  $(1/(1 - y_i))$ . The latter could also be used on the sliderule. Its use would involve no mathematical approximation and should be no more difficult to calculate. Correspondence with the American Gas and Electric Service Company indicates that this system is now (while this article is in proof) using this exact penalty factor method.



transportation costs will average about \$150,000 per year. These benefits are apparently sufficiently large to justify the costs of the additional computations.

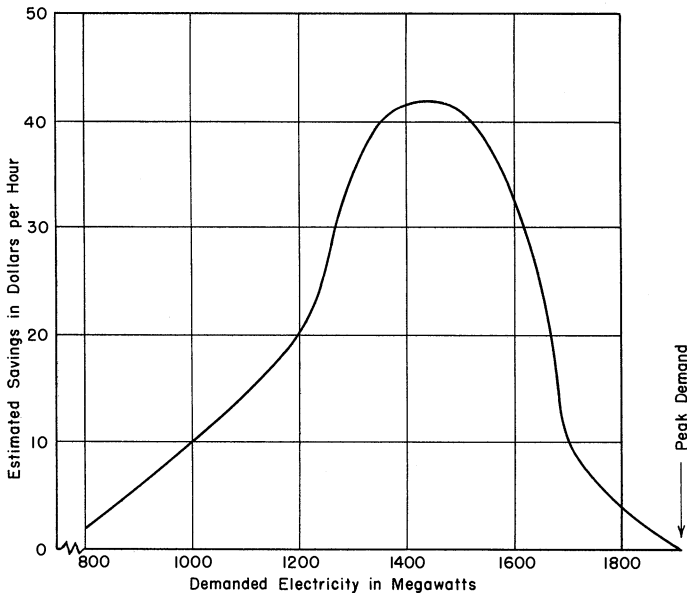


FIGURE IV

Cost reductions from taking account of transmission losses in model of American Gas and Electric Company System.

Source: Kirchmayer and Stagg, *op. cit.*, p. 519.

## VI

Fictitious illustrations of the theoretical constructs of classical economics abound; authentic applications and tests are rare. In these pages we have come to grips with a specific problem in marginal analysis that has a very real counterpart in the day-to-day management of firms in the electric power industry.

We proceeded within the framework of the classical model of the profit motivated firm. Using the hypothesis of cost minimization we were able to specify how such a model firm would apportion production among the several plants under its control. Two cases were distinguished. For each, the model by any reasonable standards proved itself not only as a predictive device, but also showed itself of fundamental importance for solving the problem in practice.

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