



Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS

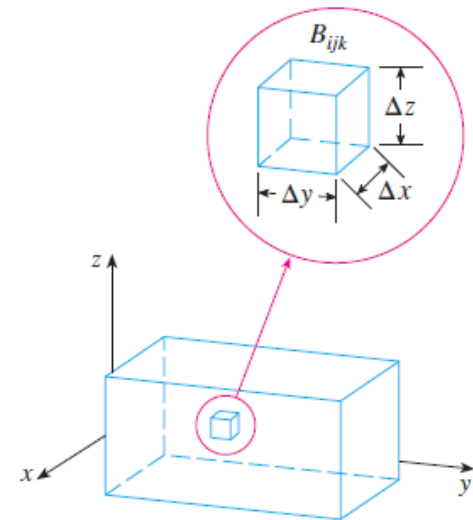
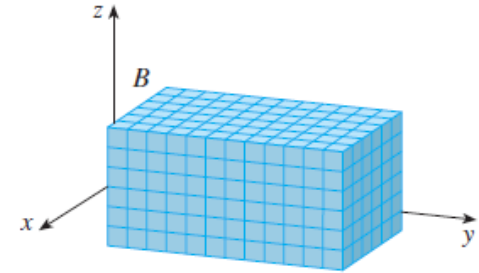
MATH 1110

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TRIPLE INTEGRAL

$$\iiint_B f(x, y, z) \, dV = \lim_{l, m, n \rightarrow \infty} \sum_{l=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

$$\iiint_B f(x, y, z) \, dV = \int_r^s \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz$$



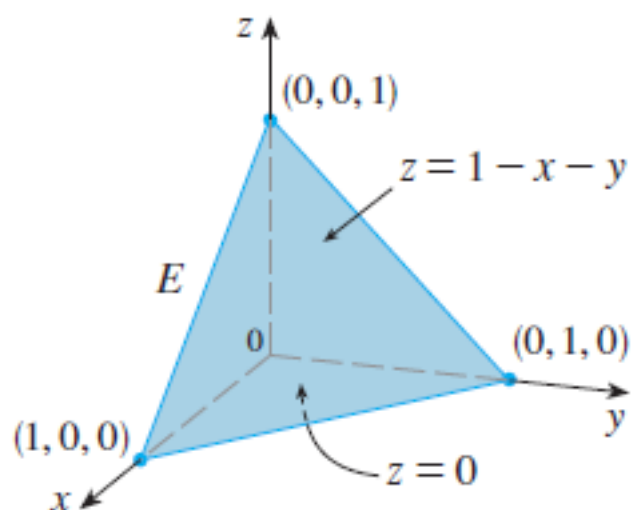
EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2 dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$$

SOLUTION We could use any of the six possible orders of integration. If we choose to integrate with respect to x , then y , and then z , we obtain

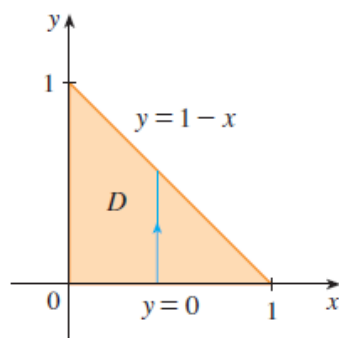
$$\begin{aligned}\iiint_B xyz^2 dV &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dx dy dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2 y z^2}{2} \right]_{x=0}^{x=1} dy dz \\&= \int_0^3 \int_{-1}^2 \frac{y z^2}{2} dy dz = \int_0^3 \left[\frac{y^2 z^2}{4} \right]_{y=-1}^{y=2} dz \\&= \int_0^3 \frac{3z^2}{4} dz = \left[\frac{z^3}{4} \right]_0^3 = \frac{27}{4}\end{aligned}$$

EXAMPLE 2 Evaluate $\iiint_E z \, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.



SOLUTION

$$E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}$$



$$\begin{aligned} \iiint_E z \, dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2} \right]_{z=0}^{z=1-x-y} dy \, dx \\ &= \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2 dy \, dx = \frac{1}{2} \int_0^1 \left[-\frac{(1-x-y)^3}{3} \right]_{y=0}^{y=1-x} dx \\ &= \frac{1}{6} \int_0^1 (1-x)^3 dx = \frac{1}{6} \left[-\frac{(1-x)^4}{4} \right]_0^1 = \frac{1}{24} \end{aligned}$$

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2} \, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

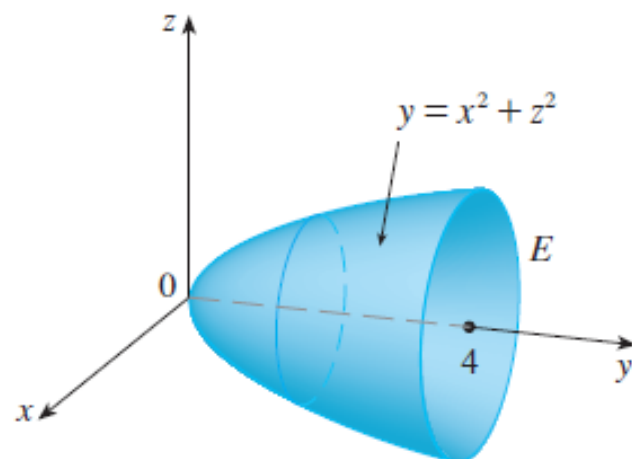


FIGURE 9

Region of integration

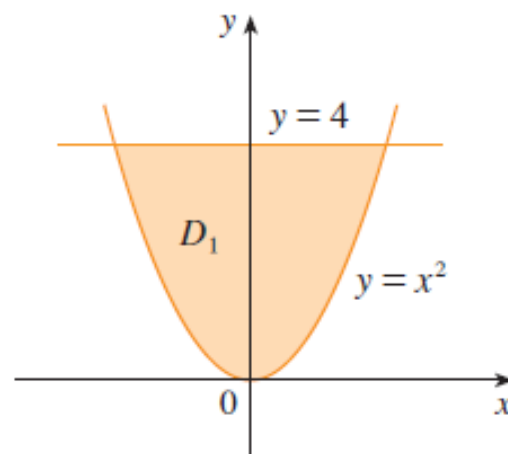


FIGURE 10

Projection onto xy -plane

SOLUTION

From $y = x^2 + z^2$ we obtain $z = \pm\sqrt{y - x^2}$, so the lower boundary surface of E is $z = -\sqrt{y - x^2}$ and the upper surface is $z = \sqrt{y - x^2}$. Therefore the description of E as a type 1 region is

$$E = \{(x, y, z) \mid -2 \leq x \leq 2, x^2 \leq y \leq 4, -\sqrt{y - x^2} \leq z \leq \sqrt{y - x^2}\}$$

and so we obtain

$$\iiint_E \sqrt{x^2 + z^2} \, dV = \int_{-2}^2 \int_{x^2}^4 \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2} \, dz \, dy \, dx$$

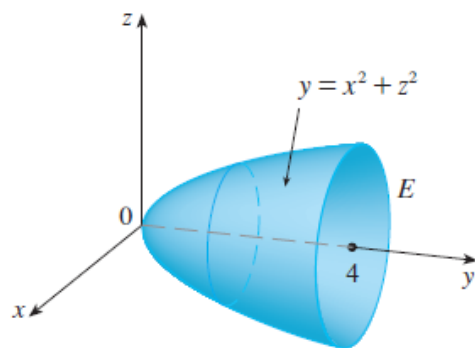


FIGURE 9
Region of integration

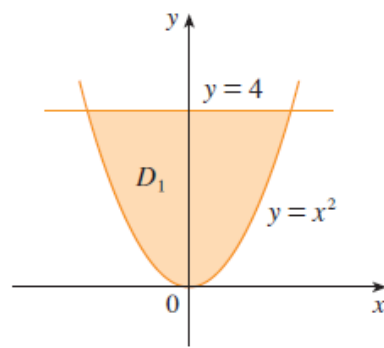


FIGURE 10
Projection onto xy -plane

Although this expression is correct, it is extremely difficult to evaluate. So let's instead consider E as a type 3 region. As such, its projection D_3 onto the xz -plane is the disk $x^2 + z^2 \leq 4$ shown in Figure 11.

Then the left boundary of E is the paraboloid $y = x^2 + z^2$ and the right boundary is the plane $y = 4$, so taking $u_1(x, z) = x^2 + z^2$ and $u_2(x, z) = 4$ in Equation 11, we have

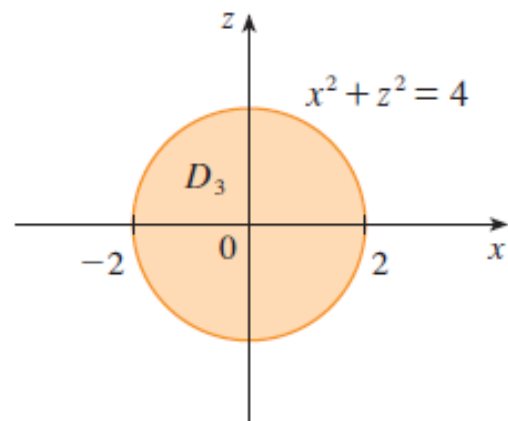
$$\iiint_E \sqrt{x^2 + z^2} \, dV = \iint_{D_3} \left[\int_{x^2+z^2}^4 \sqrt{x^2 + z^2} \, dy \right] dA = \iint_{D_3} (4 - x^2 - z^2) \sqrt{x^2 + z^2} \, dA$$

Although this integral could be written as

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2) \sqrt{x^2 + z^2} \, dz \, dx$$

it's easier to convert to polar coordinates in the xz -plane: $x = r \cos \theta$, $z = r \sin \theta$. This gives

$$\begin{aligned} \iiint_E \sqrt{x^2 + z^2} \, dV &= \iint_{D_3} (4 - x^2 - z^2) \sqrt{x^2 + z^2} \, dA \\ &= \int_0^{2\pi} \int_0^2 (4 - r^2) r \, r \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4) \, dr \\ &= 2\pi \left[\frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 = \frac{128\pi}{15} \end{aligned}$$



EXAMPLE 4:

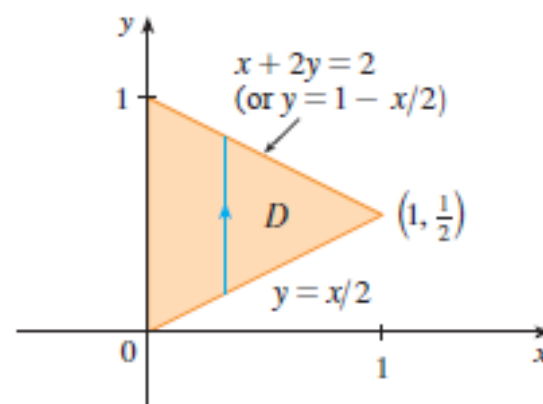
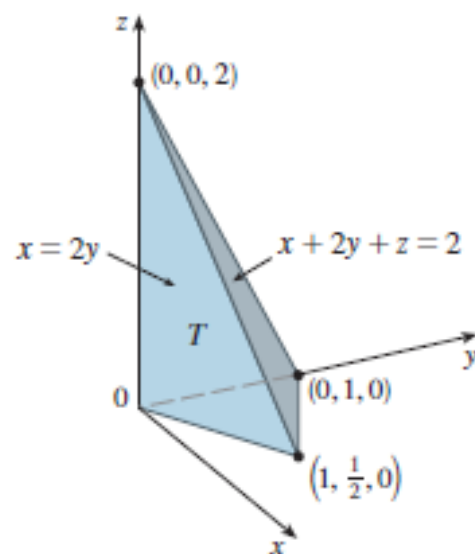
Evaluate $\iiint_Q 3xy^3 z^2 dV$ if

$$Q = \{(x, y, z): -1 \leq x \leq 3, 1 \leq y \leq 4, 0 \leq z \leq 2\}.$$

Solution Of the six possible iterated integrals, we shall use the following:

$$\begin{aligned}\int_1^4 \int_{-1}^3 \int_0^2 3xy^3 z^2 dz dx dy &= \int_1^4 \int_{-1}^3 xy^3 z^3 \Big|_0^2 dx dy \\ &= \int_1^4 \int_{-1}^3 8xy^3 dx dy = \int_1^4 4x^2 y^3 \Big|_{-1}^3 dy \\ &= \int_1^4 (36y^3 - 4y^3) dy = 8y^4 \Big|_1^4 = 2040.\end{aligned}$$

EXAMPLE 5 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.



SOLUTION The tetrahedron T and its projection D onto the xy -plane are shown in Figures 14 and 15. The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.

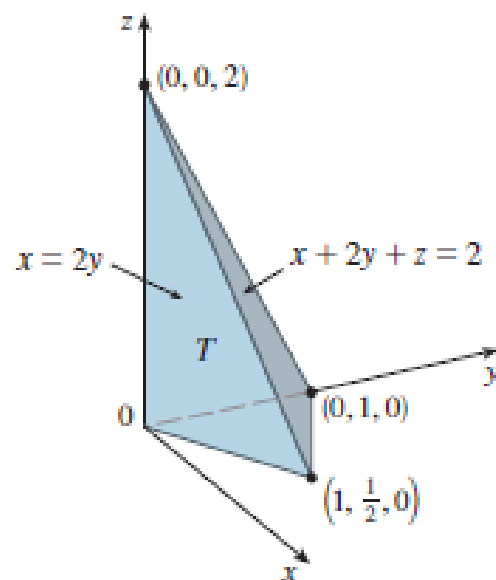


FIGURE 14

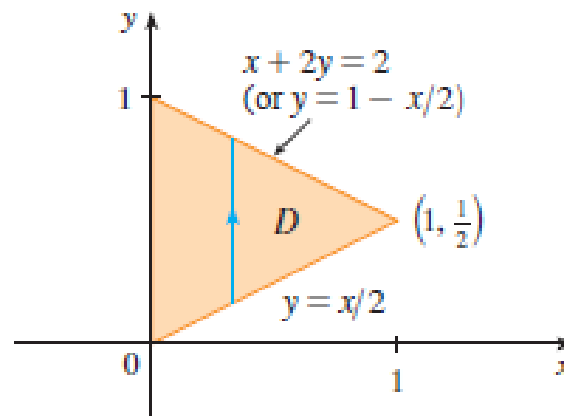


FIGURE 15

Therefore we have

$$\begin{aligned} V(T) &= \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz \, dy \, dx \\ &= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y) \, dy \, dx = \frac{1}{3} \end{aligned}$$

