



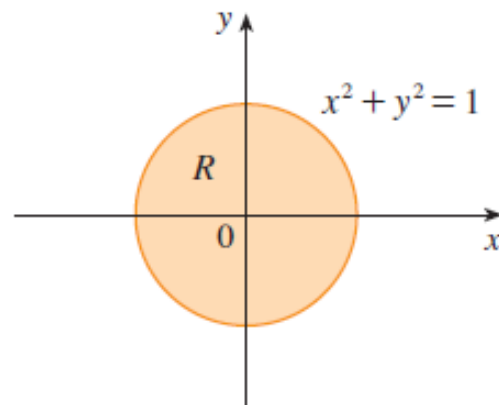
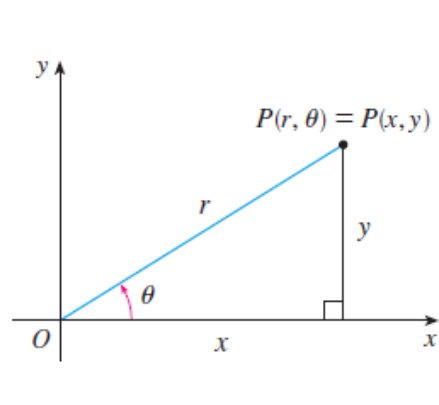
Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS

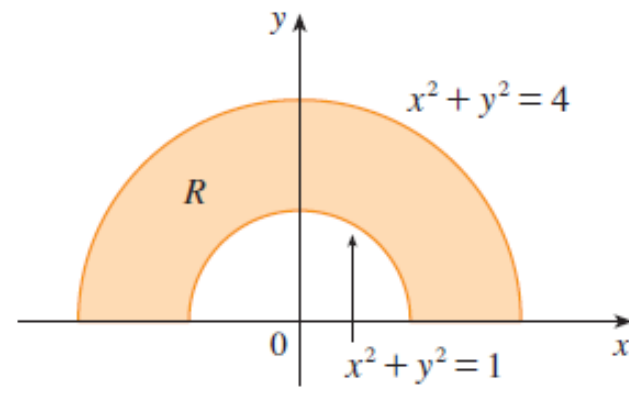
MATH 1110

: Instructor
Dr. O. Philips Agboola
pagboola@ksu.edu.sa
Office: F054

Double INTEGRAL IN POLAR COORDINATES



(a) $R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$



(b) $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

The rectangular coordinates (x, y) and polar coordinates (r, θ) of a point P are related as follows:

(i) $x = r \cos \theta, \quad y = r \sin \theta$

(ii) $\tan \theta = \frac{y}{x}, \quad r^2 = x^2 + y^2$

Double INTEGRAL IN POLAR COORDINATES

$$\iint_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta$$

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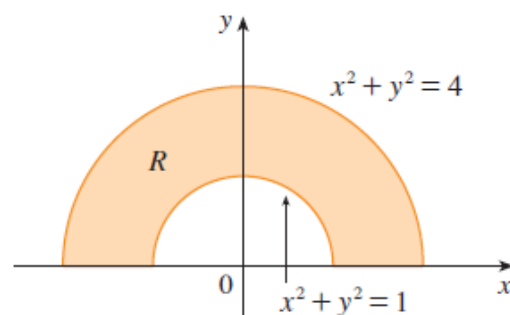
EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2) dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

SOLUTION The region R can be described as

$$R = \{(x, y) \mid y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$$

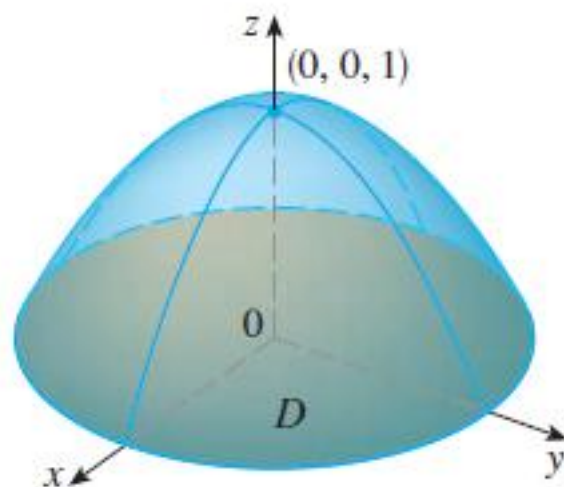
It is the half-ring shown in Figure 1(b), and in polar coordinates it is given by $1 \leq r \leq 2$, $0 \leq \theta \leq \pi$. Therefore, by Formula 2,

$$\begin{aligned} \iint_R (3x + 4y^2) dA &= \int_0^\pi \int_1^2 (3r \cos \theta + 4r^2 \sin^2 \theta) r dr d\theta \\ &= \int_0^\pi \int_1^2 (3r^2 \cos \theta + 4r^3 \sin^2 \theta) dr d\theta \\ &= \int_0^\pi \left[r^3 \cos \theta + r^4 \sin^2 \theta \right]_{r=1}^{r=2} d\theta = \int_0^\pi (7 \cos \theta + 15 \sin^2 \theta) d\theta \\ &= \int_0^\pi \left[7 \cos \theta + \frac{15}{2}(1 - \cos 2\theta) \right] d\theta \\ &= 7 \sin \theta + \frac{15\theta}{2} - \frac{15}{4} \sin 2\theta \Big|_0^\pi = \frac{15\pi}{2} \end{aligned}$$



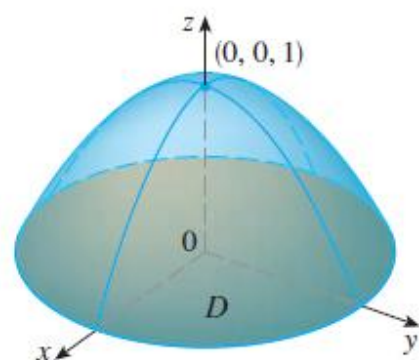
(b) $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

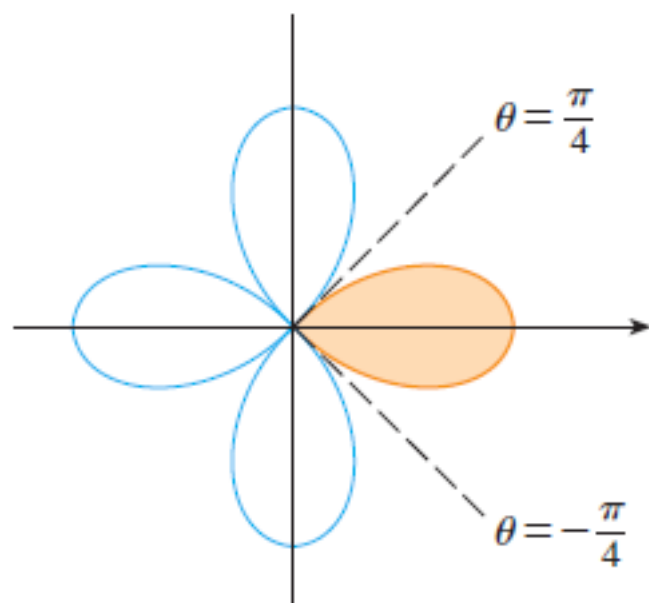


SOLUTION If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \leq 1$ [see Figures 6 and 1(a)]. In polar coordinates D is given by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, the volume is

$$\begin{aligned} V &= \iint_D (1 - x^2 - y^2) dA = \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 (r - r^3) dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2} \end{aligned}$$



EXAMPLE 3 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



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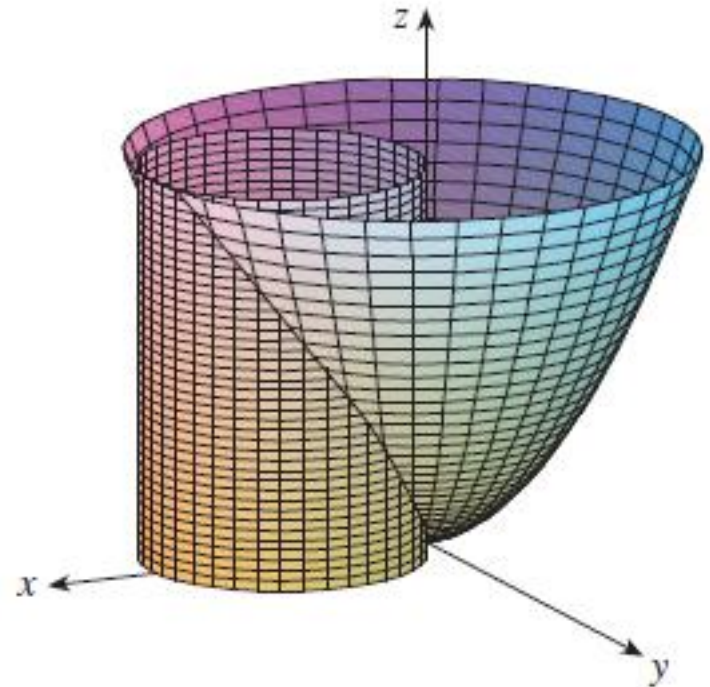
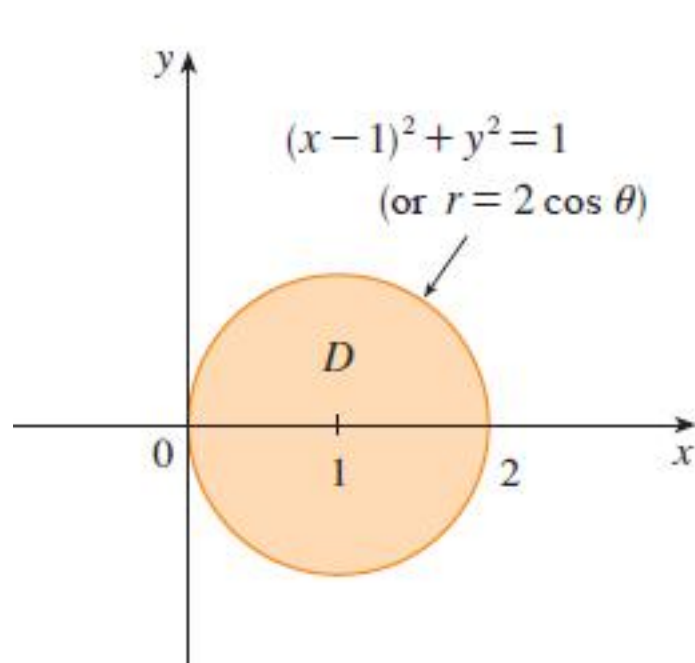
SOLUTION From the sketch of the curve in Figure 8, we see that a loop is given by the region

$$D = \{(r, \theta) \mid -\pi/4 \leq \theta \leq \pi/4, 0 \leq r \leq \cos 2\theta\}$$

So the area is

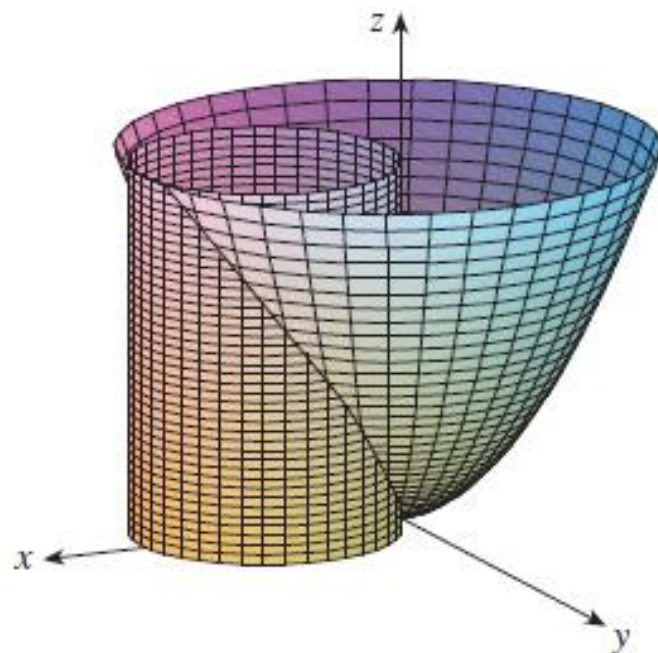
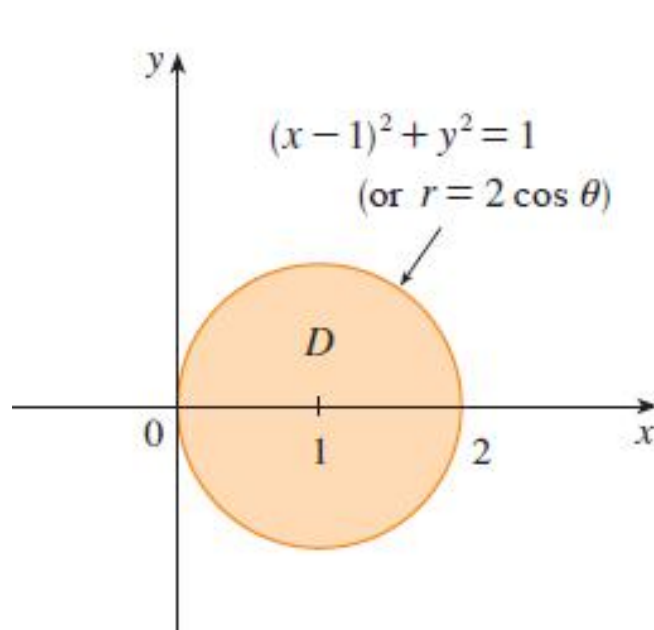
$$\begin{aligned} A(D) &= \iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r \, dr \, d\theta \\ &= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2} r^2 \right]_0^{\cos 2\theta} d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta \\ &= \frac{1}{4} \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta) \, d\theta = \frac{1}{4} \left[\theta + \frac{1}{4} \sin 4\theta \right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8} \end{aligned}$$

EXAMPLE 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy -plane, and inside the cylinder $x^2 + y^2 = 2x$.



SOLUTION The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square,

$$(x - 1)^2 + y^2 = 1$$



In polar coordinates we have $x^2 + y^2 = r^2$ and $x = r \cos \theta$, so the boundary circle becomes $r^2 = 2r \cos \theta$, or $r = 2 \cos \theta$. Thus the disk D is given by

$$D = \{(r, \theta) \mid -\pi/2 \leq \theta \leq \pi/2, 0 \leq r \leq 2 \cos \theta\}$$

and, by Formula 3, we have

$$\begin{aligned} V &= \iint_D (x^2 + y^2) dA = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r^2 r dr d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \right]_0^{2 \cos \theta} d\theta \\ &= 4 \int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta = 8 \int_0^{\pi/2} \cos^4 \theta d\theta = 8 \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right)^2 d\theta \\ &= 2 \int_0^{\pi/2} [1 + 2 \cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)] d\theta \\ &= 2 \left[\frac{3}{2} \theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right]_0^{\pi/2} = 2 \left(\frac{3}{2} \right) \left(\frac{\pi}{2} \right) = \frac{3\pi}{2} \end{aligned}$$