



Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS

MATH 1110

: Instructor
Dr. O. Philips Agboola
pagboola@ksu.edu.sa
Office: F054

Applications of Double Integrals

Density and Mass

Consider a lamina with variable density. Suppose the lamina occupies a region of the xy -plane and its **density** (in units of mass per unit area) at a point in D is given by $\rho(x, y)$, where ρ is a continuous function on D . This means that

$$\rho(x, y) = \lim_{\Delta A \rightarrow 0} \frac{\Delta m}{\Delta A}$$

where Δm and ΔA are the mass and area of a small rectangle that contains x, y and the limit is taken as the dimensions of the rectangle approach 0. (See Figure 1.)

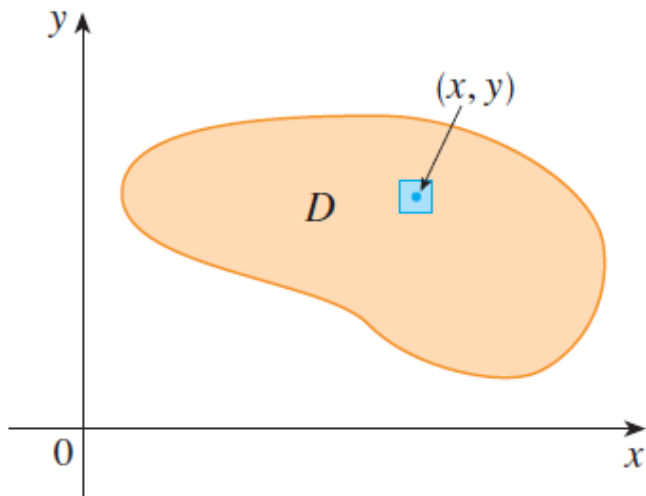
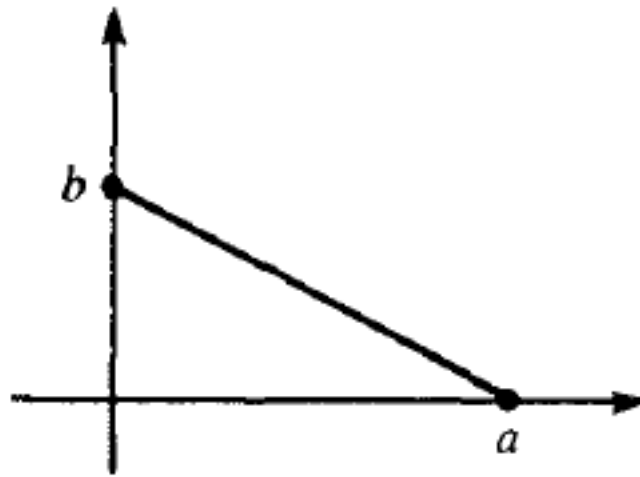


FIGURE 1

$$m = \iint_D \rho(x, y) \, dA$$

EXAMPLE 1:

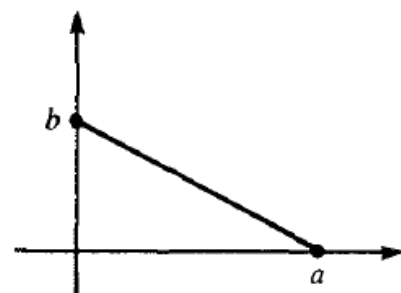
Find the mass of a plate in the form of a right triangle with legs a and b , if the density (mass per unit area) is numerically equal to the sum of the distances from the legs.



The equation of a straight line is usually written this way:

$$y = mx + b$$

SOLUTION 1:



The density $\delta(x, y) = x + y$.

$$m = \iint_D \rho(x, y) \, dA$$

$$= \int_0^a \int_0^{b-(b/a)x} (x + y) \, dy \, dx$$

$$\int_0^a \left(xy + \frac{1}{2} y^2 \right) \Big|_0^{b-(b/a)x} \, dx$$

$$= \frac{b}{a} \int_0^a (a - x) \left[x + \frac{1}{2} (b/a)(a - x) \right] \, dx$$

$$\begin{aligned}
&= \frac{b}{a} \int_0^a [ax - x^2 + \frac{1}{2}(b/a)(a-x)^2] dx = \\
&\quad (b/a) \left[\frac{1}{2}ax^2 - \frac{1}{3}x^3 + \frac{1}{2}(b/a) \cdot \frac{1}{3}(a-x)^3 \right]_0^a \\
&= (b/a) \left\{ \left(\frac{1}{2}a^3 - \frac{1}{3}a^3 \right) - \left[-\frac{1}{2}(b/a) \cdot \frac{1}{3}a^3 \right] \right\} = \\
&\quad (b/a) \left(\frac{1}{6}a^3 + \frac{1}{6}ba^2 \right) = \frac{1}{6}ba(a+b).
\end{aligned}$$

EXAMPLE 2:

Find the mass of a circular plate \mathcal{R} of radius a whose density is numerically equal to the distance from the center.

Let the circle be $r = a$.

$$\text{Then } M = \iint_{\mathcal{R}} r \, dA = \int_0^{2\pi} \int_0^a r \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{3} r^3 \right]_0^a d\theta = \int_0^{2\pi} \frac{1}{3} a^3 \, d\theta =$$

$$\frac{1}{3} a^3 \cdot 2\pi = \frac{2}{3} \pi a^3.$$

EXAMPLE 3:

Find the mass of a solid right circular cylinder \mathcal{R} of height h and radius of base b , if the density (mass per unit volume) is numerically equal to the square of the distance from the axis of the cylinder.

$$M = \iiint \rho \, dV =$$

$$\int_0^{2\pi} \int_0^b \int_0^h r^2 \cdot r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^b r^3 h \, dr = \int_0^{2\pi} \left[\frac{1}{4} h r^4 \right]_0^b d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} h b^4 \, d\theta$$

$$= \frac{1}{4} h b^4 \cdot 2\pi = \frac{1}{2} \pi h b^4.$$